Solutions to the Third Annual Columbus State Calculus Pre-calculus contest

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1. Given two positive real numbers x and y such that $\ln(x^2y^3) = 1$ and $\ln(x^5y^7) = 1$, find $\ln(xy^3)$.

   (A) 1 (B) 2 (C) 3 (D) 4 (E) $\boxed{5}$

   Solution: Multiplying the first equation by 8, the second by 3 and subtracting the two new ones, gives
   $$\ln \frac{x^{16}y^{24}}{x^{15}y^{21}} = \ln(xy^3) = 8 - 3 = 5.$$
   Hence, the answer is E. ■

2. The equation $3^{3x-3} + 3^{2x-2} + 3^{x-2} = 1$ has only one real solution, say x_0. The solution can be written $x_0 = \log_3[(10m+n)^{1/3} - 1]$ for some natural numbers m and n less than 10. What is $n - 3m$?

   (A) 1 (B) $\boxed{2}$ (C) 3 (D) 4 (E) 5

   Solution: If we set $3^x = t$ and multiplying by $3^3 = 27$ the given equation, we obtain $t^3 + 3t^2 + 3t = 27$ which implies $(t+1)^3 = 28$. Thus, the solution in t is $t = 28^{1/3} - 1$ and for $x_0 = \log_3[28^{1/3} - 1]$. This shows that $m = 2$ and $n = 8$. So, B is the correct answer. ■

3. If a and b are real numbers such that $a > b > 0$, and
   $$\frac{a^3 - b^3}{(a-b)^3} = 4,$$
   then the ratio $\frac{a}{b}$ can be written as $\frac{a}{b} = \frac{m+\sqrt{n}}{2}$ for some natural numbers m and n. Find $n - m$.

   (A) 1 (B) $\boxed{2}$ (C) 3 (D) 4 (E) 5

   Solution: We simplify the given fraction by $a - b > 0$ and obtain $a^2 + ab + b^2 = 4(a^2 - 2ab + b^2)$. This gives the equation in $a/b = t$: $t^2 - 3t + 1 = 0$. Using the quadratic formula we obtain $t = \frac{3+\sqrt{5}}{2}$. Hence we obtain $n - m = 2$ and so the correct answer is B. ■

4. The remainder of the division $\left(\frac{5-2x^2}{3}\right)^{2015} \div (x^2 - x - 2)$ is of the form $A + Bx$. Find $A - 4B$.

   (A) 1 (B) 2 (C) $\boxed{3}$ (D) 4 (E) 5

Solution: We observe first that $x^2 - x - 2 = (x+1)(x-2)$. So, using Factor Theorem, we must have $A + B(-1) = P(-1)$ and $A + 2B = P(2)$ where $P(x) = \left(\frac{5-2x^2}{3}\right)^{2015}$. Hence $A - B = 1$ and $A + 2B = -1$. Solving this system in terms of A and B, we get $B = -2/3$ and $A = 1/3$. This means that we get $A - 4B = 3$, which gives the answer C. ■

5. [$*^2$] The quartic equation $x^4 + 2x^3 - 10x^2 - 2x + 1 = 0$ has four (complex) solutions of the form, $x_1 = 1 + \sqrt{n}$ and $x_2 = 1 - \sqrt{n}$, $x_3 = p + \sqrt{m}$ and $x_4 = p - \sqrt{m}$ for some integers number m, n and p. Find $m - 2n$.

   (A) $\boxed{1}$ (B) 2 (C) 3 (D) 4 (E) 5

Solution: We observe that $x_1 + x_2 = 2$, which implies that $x_3 + x_4 = -2 - 2 = -4 = -2p$ by Viete's relations. Then the equation can be factor as $(x^2 - 2x + a)(x^2 + 4x + b) = x^3 + x^2 + (a + b - 8)x^2 + (4a - 2b)x + ab$. This gives the linear system $a + b = -2$ and $4a - 2b = -2$. Solving for a and b we get $a = b = -1$. Hence $x_1x_2 = 1 - n = a = -1$ which means that $n = 2$ and $x_3x_4 = p^2 - m = 4 - m = -1$ which gives $m = 5$. So, the answer is A. ■

6. [$*^5$] For x in the interval $(6, 11)$ the equation in t,

$$|t + 4| - |t - 2| + |t - 7| = x$$

has four solutions written in increasing order, $t_1 < t_2 < t_3 < t_4$. The expression $t_3t_4 - t_1t_2$ can be simplified to $mx + n$ for some whole numbers m and n. Find the value of m.

   (A) 1 (B) $\boxed{2}$ (C) 3 (D) 4 (E) 5

Solution: For $t \leq -4$ the equation becomes $-t - 4 - (2 - t) + 7 - t = x$ or $t = 1 - x$. We observe that for $x \in (6, 11)$, $t = 1 - x < 1 - 6 = -5 < -4$ so we get a valid solution in this case.

For $t \in (-4, 2]$, the equation becomes $t + 4 - (2 - t) + 7 - t = x$. This gives $t = x - 9$. Again, since $x \in (6, 11)$, then $t = x - 9 \in (-3, 2)$ which shows that this is a valid solution.

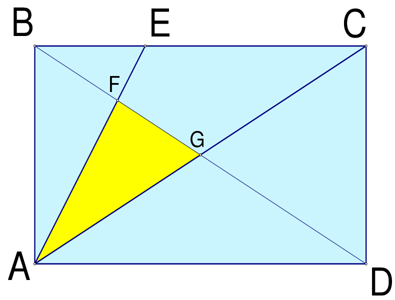
For $t \in (2, 7]$, the equation becomes $t + 4 - (t - 2) + 7 - t = x$ or $t = 13 - x$. For $x \in (6, 11)$, then $t \in (2, 7)$ which is what is suppose to be.

Finally, if $t > 7$, we obtain $t = x+1$. Because of our analysis we observe that $t_1 = 1-x$, $t_2 = x - 9$, $t_3 = 13 - x$ and $t_4 = x + 1$. Hence,

$$t_3 t_4 - t_1 t_2 = (13 - x)(x + 1) - (1 - x)(x - 9) = 22 + 2x$$

which leads to the answer B. ■

7. In the accompanying figure we have a rectangle $ABCD$ and G the intersection of its diagonals. We know that $\overline{AE}$ and $\overline{AC}$ are trisecting the angle $\angle BAD$. Let F be the intersection of the diagonal $\overline{BD}$ and $\overline{AE}$. Knowing that $AB = 1$ then what is the ratio $\frac{Area(ABCD)}{2Area(AFG)}$?

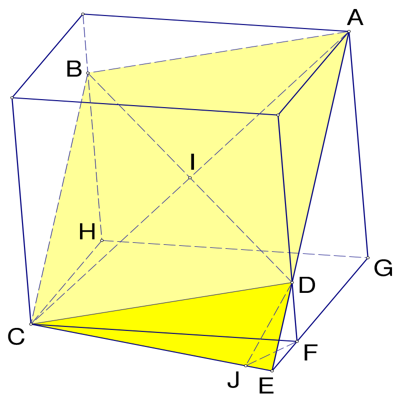


(A) 1 (B) 2 (C) 3

(D) $\boxed{4}$ (E) 5

Solution: Angles $\angle BAE$, $\angle EAC$, and $\angle CAD$ are all equal to $30°$ and so the triangle $\triangle ABG$ is equilateral and then $\triangle FAG$ has half the area of $\triangle ABG$. Hence, $2Area(AFG) = Area(ABG) = (1/2)Area(ABD) = (1/4)Area(ABCD)$. Therefore, the answer is D. ■

8. $[*^3]$ In the accompanying figure we have a section ABCD into a cube of side-lengths 1, which cuts the cube along the diagonal $\overline{AC}$, and points B and D divide the respective sides into ratios (top to bottom) $1 : 2$ and $2 : 1$. What is the area of $ABCD$?



(A) $\frac{3\sqrt{13}}{8}$ (B) $\frac{\sqrt{14}}{4}$ (C) $\boxed{\frac{\sqrt{14}}{3}}$

(D) $\frac{\sqrt{15}}{3}$ (E) $\frac{\sqrt{13}}{3}$

Solution: Method I First we observe that $ABCD$ is a parallelogram. Hence it is enough to calculate the area of the triangle ACD. Let us extend $\overline{AD}$ (see figure above) until intersects the plane of the bottom face of the cube $FGHC$. Then from similarity of triangles DEF and AEG we calculate that $EF = 1/2$ and observe that $\frac{AD}{DE} = \frac{FG}{EF} = 2$. This shows that $Area(ACD) = 2Area(CDE)$. Therefore, we just need to calculate the area of $\triangle CDE$. Using Pythagorean Theorem in the triangle EFC we have $CE = \frac{\sqrt{5}}{2}$. If J is the foot of the altitude corresponding to base $\overline{CE}$ in the triangle EFC, we obtain $JF = \frac{1}{\sqrt{5}}$. $\overline{DJ}$ is the altitude in the triangle $\triangle CDE$ corresponding to base $\overline{CE}$ and using Pythagorean Theorem again we have $DJ = \sqrt{(1/3)^2 + (1/5)} = \frac{\sqrt{14}}{3\sqrt{5}}$. This implies that the area of $\triangle CDE$ is $\frac{1}{2}\frac{\sqrt{5}}{2}\frac{\sqrt{14}}{3\sqrt{5}}$ and so $Area(ABCD) = 4Area(CDE) = \frac{\sqrt{14}}{3}$. Answer: C.

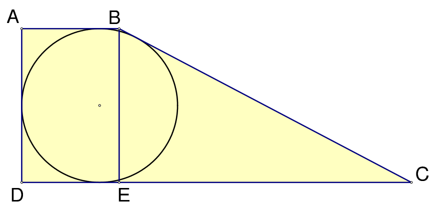
Method II. One can use Heron's Formula to compute the area of the triangle ACD since $AC = \sqrt{3}$, $AD = BC = \sqrt{13}/3$ and $CD = \sqrt{10}/3$. In this case it is useful to use a different version of Heron's Formula:

$$A = \frac{1}{4}\sqrt{2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)},\ or$$

$$A = \frac{1}{4}\sqrt{2a^2(b^2 + c^2) - a^4 - (b^2 - c^2)^2}.$$

We observe that we can set $a^2 = 3$, $b^2 = 13/9$ and $c^2 = 10/9$ in the above formula. This gives $A^2 = \frac{1}{16}[(2(3)(23/9) - 9 - (1/9)] = \frac{7}{18}$. Therefore $Area(ABCD) = 2A = \frac{\sqrt{14}}{3}$. ■

9. [$*^2$] In the trapezoid ABCD with bases $\overline{AB}$ and $\overline{DC}$, two of its side lengths are $AD = 8$ and $BC = 17$. Knowing that $m(\angle ADC) = 90°$ and the trapezoid $ABCD$ is cyclic (there exists a circle tangent to all sides-see figure on the right), find the area of ABCD.



(A) 60 (B) 70 (C) 80

(D) 90 (E) $\boxed{100}$

Solution: We let E on $\overline{DC}$ (as shown) so that $\overline{BE}$ is perpendicular to $\overline{DC}$. In the right triangle $\triangle BEC$ we find easily $EC = 15$ (remember the Pythagorean triple 8, 15 and 17). Hence, if we denote the base $AB = x$, we have the equation $x+x+15 = 8+17$ (the opposite sides in a cyclic quadrilateral add up to the same number). Therefore $x = 5$ and so the area is $\frac{(5+20)8}{2} = 100$, the answer is E. ■

10. [$*^1$] The trigonometric equation

$$8\cos(x)\cos(2x)\cos(4x) = 1$$

has exactly three solutions in the interval $(0, \frac{\pi}{2})$ (measured in radians). If we write these three solutions in increasing order, $x_1 < x_2 < x_3$, then the expression $\frac{x_1+x_2-x_3}{\pi}$

is a positive rational number which in reduced form, $\frac{a}{b}$, leads to what number at the numerator?

(A) 1 (B) 2 (C) 3 (D) $\boxed{4}$ (E) 5

Solution: If we multiply the equation by $\sin(x)$ (not equal to zero in the interval $(0, \frac{\pi}{2})$), and using the double angle formula $2\sin\alpha\cos\alpha = \sin(2\alpha)$, we get an equivalent equation

$$\sin(8x) = \sin x,$$

which attracts $8x = x + 2k\pi$ or $8x = (\pi - x) + 2k\pi$, for some integer k. We get $\frac{2\pi}{7}$, $\frac{\pi}{9}$ and $\frac{\pi}{3}$ the three solutions in $(0, \frac{\pi}{2})$. Since $\frac{1}{9} < \frac{2}{7} < \frac{1}{3}$, then

$$\frac{x_1 + x_2 - x_3}{\pi} = \frac{1}{9} + \frac{2}{7} - \frac{1}{3} = \frac{2}{7} - \frac{2}{9} = \frac{4}{63}.$$

This implies that the correct answer is D. ■