Solutions to the Second Annual Columbus State Calculus Pre-contest

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Pre-calculus Problems

1. Find the remainder of the division $(x^5 - 5x^4 + 10x^3 - 9x^2 + 5x - 1) \div (x^2 - 2x + 1)$

   (A) $2x - 3$ (B) $1 - 2x$ (C) $3x - 2$ (D) $2 - x$ (E) $2x - 1$

   Solution: We observe first that $(x-1)^2 = x^2 - 2x + 1$ and one can check that $(x-1)^5 = x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$ (or by looking at the Pascal's triangle). Hence,

   $$x^5 - 5x^4 + 10x^3 - 9x^2 + 5x - 1 = (x-1)^2(x-1)^3 + x^2,$$

   or

   $$x^5 - 5x^4 + 10x^3 - 9x^2 + 5x - 1 = (x-1)^2[(x-1)^3 + 1] + 2x - 1,$$

   which gives the answer E. ■

2. For some positive numbers a and b we have the identity

   $$\frac{\cos 3x}{\sin 5x} - \frac{\sin 3x}{\cos 5x} = a[\sin(2x) + \cos(2x)\cot(bx)], \quad x \in (0, \frac{\pi}{20}).$$

   What is $\frac{b}{a}$?

   (A) 4 (B) 5 (C) 6 (D) 7 (E) 8

   Solution: If we bring to the same denominator, the two fractions, and use the addition formula for *cosine* we get

   $$E := \frac{\cos 3x}{\sin 5x} - \frac{\sin 3x}{\cos 5x} = \frac{\cos 5x \cos 3x - \sin 5x \sin 3x}{\sin 5x \cos 5x} = \frac{\cos(5x + 3x)}{\sin 5x \cos 5x}.$$

We can continue, using the double angle formula $\sin 2\alpha = 2\sin\alpha\cos\alpha$

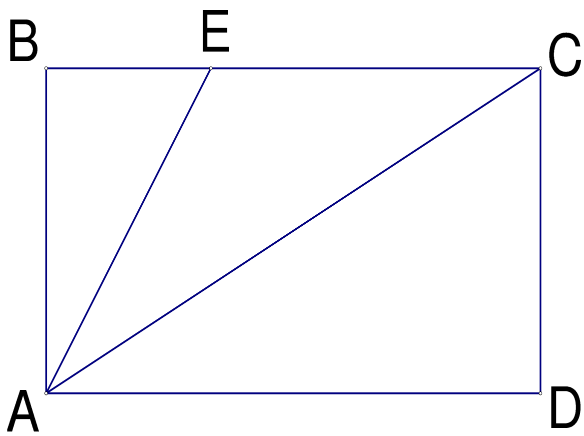
$$E = \frac{\cos 8x}{\sin 5x \cos 5x} = 2\frac{\cos(10x - 2x)}{\sin 10x} = 2(\cos 2x \cot 10x + \sin 2x),$$

which gives $a = 2$ and $b = 10$. Then we obtain $b/a = 5$ and so B is the correct answer. ■

3. [*[1]] In the accompanying figure we have a rectangle ABCD with E on $\overline{BC}$ such that $\overline{AE}$ and $\overline{AC}$ are trisecting the angle $\angle BAD$. Knowing that $AB = 1$ then what is the value of EC?

   (A) $\frac{\sqrt{3}}{2}$ (B) $\frac{2\sqrt{3}}{3}$ (C) $\frac{\sqrt{3}}{4}$

   (D) $\frac{4\sqrt{3}}{4}$ (E) $\frac{4\sqrt{3}}{5}$



Solution: The angles $\angle BAE$, $\angle EAC$, and $\angle CAD$ are all equal to $30°$ and so the triangle $\triangle EAC$ is isosceles. Hence $EA = EC = 1/\cos 30° = 2\sqrt{3}/3$. Therefore, the answer is B ■

4. [*[5]] The cubic equation $4x^3 - x + 12 = 0$ has two solutions, x_1 and x_2, which are not real. Find $x_1 x_2$.

   (A) -1 (B) 1 (C) -2 (D) 2 (E) -3

Solution: We check to see if the given equation has any rational roots. The possible such roots are of the form a/b with b a divisor of 4 and a a divisor of 12. With a little luck one may find that $-3/2$ is indeed a zero, and so

$$4x^3 - x + 12 = (2x + 3)(2x^2 - 3x + 4)$$

which means the other two roots (which are indeed pure complex), by Viete's Relations, have a product of 2 (Answer: D). ■

5. A positive number x is 2 more than its reciprocal. Then x is in which of the intervals below?

   (A) $[(5/2, 3]$ (B) $[0, 1]$ (C) $[3, 4]$ (D) $(2, 5/2)$ (E) $(1, 1/2)$

Solution: We have the equation $x = 2 + \frac{1}{x}$ or $x^2 - 2x - 1 = 0$. Hence $(x-1)^2 = 2$ which gives $x = 1 + sqrt2$. This puts x in the interval $(2, 5/2)$. (Answer: D) ■

6. [$*^4$] If $\log_4 a = \log_{10}(b - 2a) = \log_{25} b$, what is $\frac{b}{a}$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: We set $t = \log_4 a = \log_{10}(b-2a) = \log_{25} b$ and from the first equality we see that $a = 4^t$, and then from the second equality, we have $b - 2a = 10^t$. Similarly, we get $b = 25^t$. Hence, $ab = 100^t = (b-2a)^2$. This implies the equality $b^2 - 5ab + 4a^2 = 0$ or if we denote by $x = b/a$ (note that $a \neq 0$), we get a quadratic equation in x: $x^2 - 5x + 4 = 0$ or $(x-4)(x-1) = 0$. We clearly exclude $x = 1$ and so $x = 4$. Therefore $a/b = 4$ (Answer: D). ■

7. The equation $4^x - 2^{x+1} = 24$ has one real solution, say x. Estimate in what interval is x ?

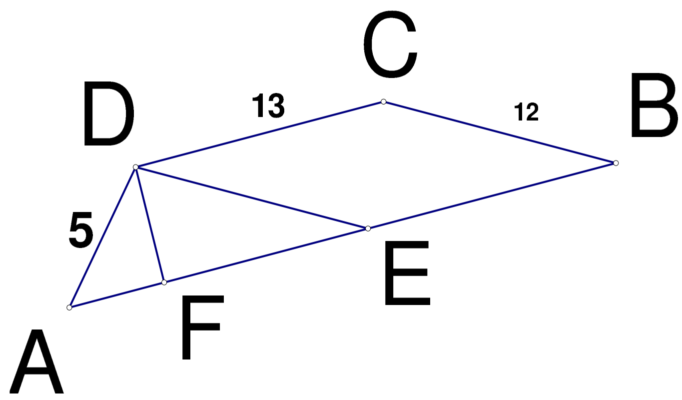
(A) $[3, 4]$ (B) $[5/2, 3]$ (C) $[1, 2]$ (D) $[2, 5/2]$ (E) $[3, 7/2]$

Solution: Let us denote by $t = 2^x$ and observe that the given equation is equivalent to $t^2 - 2t - 24 = 0$ which can be factor to $(t-6)(t+4) = 0$. We exclude the negative solution and conclude that $2^x = 6$. Since $2^{5/2} < 6$ $(32 < 36)$ and $2^3 = 8 > 6$ we see that x is in $[5/2, 3]$. So, B is the correct answer. ■

8. [$*^3$] In the trapezoid ABCD with bases $\overline{AB}$ and $\overline{DC}$, its side lengths are $AB = 26$, $BC = 12$, $DC = 13$ and $AD = 5$. What is the area of ABCD?

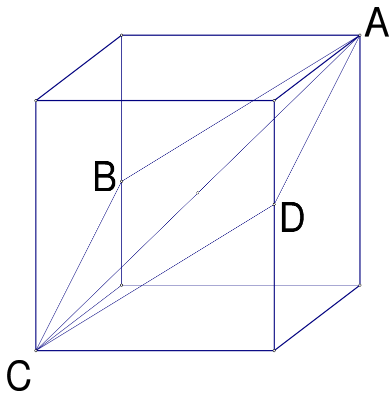
(A) 90 (B) 100 (C) 110

(D) 120 (E) 130



Solution: Let E be on $\overline{AB}$ such that $\overline{DE}$ is parallel to $\overline{BC}$. Since $DCBE$ is a parallelogram we get that $DE = 12$ and $BE = 13$. This implies that $AE = 26 - 13 = 13$. Then, the triangle ADE is a right triangle since the numbers $(5, 12, 13)$ form a Pythagorean triple. So, the area of the triangle ADE is $5(12)/2 = 30$. Then DF is equal $2(30)/13$. Then the area of $DCBE$ is $13\frac{2(30)}{13} = 60$. Hence the area of $ABCD$ is $30 + 60 = 90$. Therefore, the answer is A. ■

9. [$*^2$] In the accompanying figure we have a section ABCD into a cube of side-lengths 1, which cuts the cube along the diagonal $\overline{AC}$, and midpoints D and B of the shown sides. What is the area of $ABCD$?



(A) $\frac{\sqrt{5}}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{\sqrt{6}}{2}$

(D) $\frac{\sqrt{7}}{2}$ (E) $\frac{\sqrt{10}}{2}$

Solution: The diagonal $\overline{AC}$ is of length $\sqrt{3}$. Also the diagonal $\overline{BD}$ is of length $\sqrt{2}$. The quadrilateral $ABCD$ is a rhombus and so its area is equal to half the product of its diagonals: $\frac{\sqrt{3}\sqrt{2}}{2} = \frac{\sqrt{6}}{2}$. Therefore the answer is C. ■

10. What is the product of all real roots of the equation $x^2 + 1 = 5|x + 3|$?

(A) -10 (B) -14 (C) -16 (D) -20 (E) -6

Solution: Let us suppose that $x > -3$. Then the equation given is equivalent to $x^2 + 1 = 5x + 15$ or $x^2 - 5x - 14 = 0$. Factoring we obtain $(x - 7)(x + 2) = 0$. This gives two solutions $x_1 = 7$ or $x_2 = -2$. In the case $x < -3$, we get $x^2 + 1 = -15 - 5x$ or $x^2 + 5x + 16 = 0$ which has no real solutions. Thus, we get $7(-2) = -14$. (Answer: B).