Solutions of the Fortieth Annual Columbus State Invitational Mathematics Tournament

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1 C	6 C	11 B	16 A	21 C	26 D	31 B	36 B	41 E	46 A
2 C	7 C	12 C	17 E	22 A	27 A	32 E	37 D	42 D	47 B
3 A	8 E	13 B	18 E	23 A	28 C	33 E	38 A	43 A	48 C
4 A	9 A	14 E	19 B	24 B	29 D	34 E	39 D	44 C	49 D
5 E	10 B	15 D	20 D	25 D	30 B	35 D	40 B	45 C	50 E

1. How many positive integer divisors does 2014 have?

   (A) 2 (B) 4 (C) 6 (D) 8 (E) 10

   Answer: C

   We have $2014 = 2 \cdot 19 \cdot 53$, so the number of positive divisors is 8.

2. Which number is three times the product of its two digits?

   (A) 12 (B) 18 (C) 24 (D) 26 (E) 32

   Answer: C

   The product of the digits of 24 is 8. Three times 8 is 24.

3. $*$ If the arithmetic mean of two numbers is $10a$ and one of the numbers is $4a+5$, what is the other number?

   (A) $16a - 5$ (B) $15a - 15$ (C) $14a - 25$ (D) $13a - 15$ (E) $12a - 5$

   Answer: A

   It is easy to check that $\frac{1}{2} \cdot (4a + 5 + 16a - 5) = 10a$.

4. If x is a real number then what is the minimum possible value of $(2x-5)^2+18$?

(A) 18 (B) 16 (C) 14 (D) 12 (E) 10

Answer: A

The minimum value of $(2x-5)^2+18$ is attained for $x=\frac{5}{2}$. For any other value of x we have $(2x-5)^2>0$.

5. $*$ Which answer below is equal to $\left(\sqrt{5}^{\sqrt{5}}\right)^{\sqrt{5}}$?

(A) $5\sqrt{5}$ (B) $75\sqrt{5}$ (C) $125\sqrt{5}$ (D) $50\sqrt{5}$ (E) $25\sqrt{5}$

Answer: E

We have $\left(\sqrt{5}^{\sqrt{5}}\right)^{\sqrt{5}}=\sqrt{5}^{5}=25\sqrt{5}$.

6. For all positive real numbers a and b we define $a\star b=\frac{a}{b}+\frac{b}{a}$. Determine the value of $2\star 5$.

(A) $\frac{21}{10}$ (B) $\frac{27}{10}$ (C) $\frac{29}{10}$ (D) $\frac{33}{10}$ (E) $\frac{37}{10}$

Answer: C

$2\star 5=\frac{2}{5}+\frac{5}{2}=\frac{29}{10}$.

7. $*$ How many integers between 100 and 1000 are multiples of 11?

(A) 79 (B) 80 (C) 81 (D) 82 (E) 83

Answer: C

We need to solve the inequality $100\le 11k\le 1000$ for k integer. This implies that $10\le k\le 90$, so there are 81 possibilities.

8. Let a and b be real numbers such that $a + b = 7$ and $ab = 2$. Which of the following is the quadratic equation with roots a and b?

   (A) $x^2 - 2x + 7 = 0$ (B) $2x^2 - x + 7 = 0$ (C) $x^2 + 2x - 7 = 0$

   (D) $2x^2 - 7x + 1 = 0$ (E) $x^2 - 7x + 2 = 0$

   Answer: E

   If a and b are the solutions of the quadratic equation $x^2 + Sx + P = 0$ then we have the $a + b = -S$ and $ab = P$. This implies that $S = -7$ and $P = 2$, so the quadratic equation is $x^2 - 7x + 2 = 0$.

9. $*$ The number n is the biggest positive integer such that $4n$ is a three digit number. The number m is the smallest positive integer such that $3m$ is a three digit number. Find the number $n - m$.

   (A) 215 (B) 217 (C) 219 (D) 221 (E) 223

   Answer: A

   We have that $4n = 996$ and $3m = 102$. Therefore $n - m = 249 - 34 = 215$.

10. Find the sum of the solutions of the equation $\dfrac{1}{n-7} - \dfrac{1}{n} = \dfrac{1}{14}$.

    (A) 0 (B) 7 (C) 14 (D) 21 (E) 28

    Answer: B

    Clearly $n \neq 0$ and $n \neq 7$. Using a common denominator we obtain the equivalent equation $\dfrac{7}{(n-7)n} = \dfrac{1}{14}$. This implies that $n^2 - 7n - 98 = 0$, so $n = 14$ and $n = -7$.

11. How many pairs of two-digit positive integers (a, b) satisfy $a - b = 49$?

    (A) 40 (B) 41 (C) 49 (D) 50 (E) 51

    Answer: B

    The smallest b is 10, so $a = 59$. The biggest a is 99, so $b = 50$. Therefore $b \in \{10, 11, \ldots, 50\}$, and there are 41 possibilities.

12. $*$ When a certain solid substance is melted its volume increases by $\frac{1}{8}$. By how much does the volume decrease when the substance solidifies?

(A) $\frac{1}{7}$ (B) $\frac{1}{8}$ (C) $\frac{1}{9}$ (D) $\frac{1}{10}$ (E) $\frac{1}{11}$

Answer: C

Denote by x the volume of the solid substance. The volume of the melted solid is $\frac{9}{8}x$. Since $\frac{9}{8}x - \frac{1}{9}\left(\frac{9}{8}x\right) = x$ we get that the volume decreases by $\frac{1}{9}$.

13. $*$ The real numbers x and y satisfy the system of equations $xy - x = 25$ and $xy + y = 36$. Find $x - y$.

(A) -5 (B) -1 (C) 0 (D) 1 (E) 5

Answer: B

By subtracting the given equations we get that $x + y = 11$, so $y = 11 - x$. The first equation becomes $x(11 - x) - x = 25$. This is equivalent with $x^2 - 10x + 25 = 0$, so $x = 5$ and $y = 6$. Therefore $x - y = -1$.

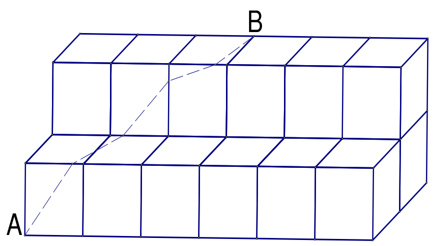
14. Which digit is in the ones' place of $8^{2013} + 9^{2014}$?

(A) 1 (B) 3 (C) 5 (D) 7 (E) 9

Answer: E

Since 2013 gives remainder 1 when divided by 4 we get that the digit in the ones' place for 8^{2013} is 8. The digit in the ones' place for 9^{2014} is 1, so the digit in the ones' place for $8^{2013} + 9^{2014}$ is 9.

15. $*$ In the adjacent figure we have 18 cubes of side-lengths 1 meter stacked as shown. A smart ant is traveling on the surface of theses cubes from point A to point B. What is the shortest distance (in meters) the ant can take?



(A) 8 (B) 7 (C) 6

(D) 5 (E) 4

Answer: D

The problem is equivalent to finding the shortest distance (in Euclidean geometry) between the opposite corners of a rectangle with sides 3 and 4. This is nothing else than the diagonal of the rectangle, so by Pitagora's theorem the shortest distance is 5.

16. Let a, b be real numbers such that $\dfrac{5a+3b}{3a+5b} = \dfrac{11}{13}$. Find $\dfrac{7a+4b}{4a+7b}$.

(A) $\frac{5}{6}$ (B) $\frac{4}{5}$ (C) $\frac{3}{4}$ (D) $\frac{2}{3}$ (E) $\frac{1}{2}$

Answer: A

By cross multiplying in the equation $\dfrac{5a+3b}{3a+5b} = \dfrac{11}{13}$ we get that $b = 2a$. This implies that $\dfrac{7a+4b}{4a+7b} = \dfrac{5}{6}$.

17. $*$ If x is a real number such that $|2014 - x| + \sqrt{x - 2015} = x$, then what is the value of $x - 2014^2$?

(A) 2011 (B) 2012 (C) 2013 (D) 2014 (E) 2015

Answer: E

Note that $x \geq 2015$, so the equation becomes $x - 2014 + \sqrt{x - 2015} = x$. This implies that$\sqrt{x - 2015} = 2014$, so $x - 2014^2 = 2015$.

18. Find the number of pairs of real numbers (a, b) which satisfy the equation $a^2 + b^2 = 2a + 2b$.

(A) 1 (B) 2 (C) 4

(D) 8 (E) infinitely many

Answer: E

The equation can be written in the form $(a-1)^2 + (b-1)^2 = 2$, so it has infinitely many solutions. Indeed, geometrically this is the equation of the circle with center $(1, 1)$ and radius $\sqrt{2}$. Since there are infinitely many points on this circle the equation has infinitely many solutions as well.

19. $*$ The perimeter of a right triangle is 16 ft. The sum of the squares of its sides is 98 ft. Find the area of the triangle.

(A) 1 ft^2 (B) 8 ft^2 (C) 16 ft^2 (D) 24 ft^2 (E) 32 ft^2

Answer: B

Denote by a, b, and c the lengths of the sides of the triangle. Then we have $a+b+c = 16$ and $a^2 + b^2 + c^2 = 98$. If c is the length of the hypotenuse then we get that $2c^2 = 98$, so $c = 7$. Therefore $a + b = 9$ and $a^2 + b^2 = 49$, so $2ab = (a + b)^2 - a^2 - b^2 = 32$. This implies that the area of the triangle is $\frac{1}{2}ab = 8$ ft^2.

20. $*$ The polynomial $19X^{2014} - 53X^{1007} + 2X^2 + 2014$ has 2014 complex roots. Find the product of all these roots.

(A) 19 (B) 38 (C) 53 (D) 106 (E) 152

Answer: D

Denote the complex roots of the polynomial by $a_1, a_2, \ldots, a_{2014}$. If the polynomial is factored as $19(X - a_1)(x - a_2) \cdots (x - a_{2014})$ then we have that $19a_1 \cdot a_2 \cdots a_{2014} = 2014$, so the product of the roots is $2014/19 = 106$.

21. $*$ Find the number of nonnegative integers n such that $\dfrac{n^2}{n+6}$ is an integer.

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

Answer: C

Note that $\dfrac{n^2}{n+6} = n - 6 + \dfrac{36}{n+6}$, so the problem is equivalent to finding all nonnegative integers n such that $n + 6$ divides 36. This implies that $n \in \{0, 3, 6, 12, 30\}$.

22. Find the sum of the integers a and b such that

$$\frac{a}{\sqrt{4+2\sqrt{3}}} + \frac{b}{\sqrt{4-2\sqrt{3}}} = \sqrt{7+4\sqrt{3}}.$$

(A) 2 (B) 4 (C) 6 (D) 8 (E) 10

Answer: A

The equation is equivalent to $\frac{a}{\sqrt{3}+1}+\frac{b}{\sqrt{3}-1}=\sqrt{3}+2$. This implies that $a(\sqrt{3}-1)+b(\sqrt{3}+1)=2(\sqrt{3}+2)$, so $(a+b)\sqrt{3}+(b-a)=2\sqrt{3}+4$. We obtain that $a+b=2$ (and $b-a=4$, so $a=-1$, $b=3$).

23. $*$ Find the number of solutions of the equation $\frac{\log_2(x-3)}{\log_2(x^2-3)}=\frac{1}{2}$.

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Answer: A

Clearly we need to have $x>3$ for the logs to be defined. Using cross multiplication and the laws of logs we obtain the equivalent equation: $\log_2(x-3)^2=\log_2(x^2-3)$. This implies that $x^2-6x+9=x^2-3$, so $x=2$. Since x needs to be greater than 3 the equation has no solutions.

24. Find the number of solutions of the equation $2^{x+3}+2^{x+2}+2^{x+1}=7^x+7^{x-1}$.

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

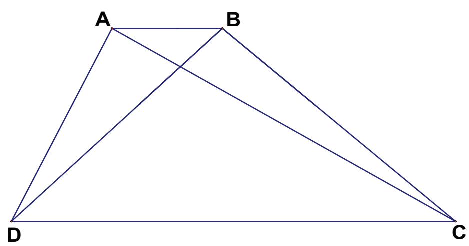
Answer: B

The equation can be written in the form $2^{x-1}(16+8+4)=7^{x-1}(7+1)$. This leads to the simplified form $7\cdot 2^{x-1}=2\cdot 7^{x-1}$, which we can write as $2^{x-2}=7^{x-2}$, so we get that $x=2$ is the only solution.

25. Let $ABCD$ be a trapezoid such that $\overline{AB}\parallel\overline{CD}$, $\overline{AC}\perp\overline{BD}$, $AC=65$ cm, and $BD=48$ cm. Find the area of $ABCD$.

(A) 195 cm^2 (B) 390 cm^2 (C) 780 cm^2

(D) 1560 cm^2 (E) 3120 cm^2



Answer: D

Denote by O the intersection of the diagonals AC and BD. The area of $\Delta ADC=\frac{AC\cdot DO}{2}$ and the area of $\Delta ABC=\frac{AC\cdot BO}{2}$. Since the area of the trapezoid $ABCD$ is the sum of the areas of these two triangles and since $DO+BO=DB$ we get that the area of $ABCD=\frac{AC\cdot BD}{2}=1560$ cm^2.

26. Let $S_n = \dfrac{1}{\sqrt{2}+1} + \dfrac{1}{\sqrt{3}+\sqrt{2}} + \cdots + \dfrac{1}{\sqrt{n+1}+\sqrt{n}}$. Determine the smallest integer n such that $S_n > 2014$.

(A) 2012^2 (B) 2013^2 (C) 2014^2 (D) 2015^2 (E) 2016^2

Answer: D

Note that S_n is a telescopic sum. Indeed, by rationalizing the denominator, we get $S_n = \sqrt{2}-1+\sqrt{3}-\sqrt{2}+\cdots+\sqrt{n+1}-\sqrt{n}$. This implies that $S_n = \sqrt{n+1}-1$. We want to find n such that $\sqrt{n+1}-1 > 2014$, so we get that $n > 2015^2 - 1$. Therefore $n = 2015^2$ is the smallest integer with the desired property.

27. $*$ Consider the function $f(x) = \ln\left(\dfrac{1+x}{1-x}\right)$ defined for all $x \in (-1,1)$. Also, let the function g defined on $(-1,1)$ by $g(x) = \dfrac{3x+x^3}{1+3x^2}$. Then the composition function $f \circ g$ can be simplified to which of the following?

(A) $3f$ (B) $2f$ (C) f (D) g (E) $2g$

Answer: A

Following the definition of the composite function we get that

$$f \circ g = \ln\left(\frac{1+\frac{3x+x^3}{1+3x^2}}{1-\frac{3x+x^3}{1+3x^2}}\right) = \ln\left(\frac{1+x}{1-x}\right)^3 = 3\ln\left(\frac{1+x}{1-x}\right) = 3f.$$

28. $*$ Let x be a real number such that $\sin x - \cos x = \dfrac{1}{2}$. Find the value of $\sin 2x$.

(A) $\dfrac{3}{8}$ (B) $\dfrac{1}{4}$ (C) $\dfrac{3}{4}$ (D) $\dfrac{5}{8}$ (E) $\dfrac{1}{2}$

Answer: C

By squaring both sides we obtain

$$\sin^2 x - 2\sin x\cos x + \cos^2 x = \frac{1}{4}.$$

Since $\sin^2 x + \cos^2 x = 1$ and $\sin 2x = 2\sin x \cos x$ we get that $\sin 2x = \dfrac{3}{4}$.

29. Let ΔABC be a triangle with sides $AB = 5$ cm, $AC = 12$ cm, and $BC = 13$ cm. Find the length of the median corresponding to $\overline{BC}$.

(A) $\frac{7}{2}$ cm (B) $\frac{9}{2}$ cm (C) $\frac{11}{2}$ cm (D) $\frac{13}{2}$ cm (E) $\frac{15}{2}$ cm

Answer: D

Note that ΔABC is a right triangle since $BC^2 = AC^2 + AB^2$ so, in particular, $\overline{BC}$ is the hypotenuse in ΔABC. Since in any right triangle the median corresponding to the hypotenuse is equal to half of the length of the hypotenuse, we get that the answer is $\frac{13}{2}$. (One can easily see this property by noticing that the median is half of the diagonal of the rectangle with sides $\overline{AB}$ and $\overline{AC}$.

30. $*$ Find the number of integers n such that $\sqrt{n^2 - 24}$ is an integer.

(A) 2 (B) 4 (C) 6 (D) 8 (E) 10

Answer: B

Note that if n is a solution then $-n$ is also a solution. Also note that $n = 0$ is not a solution of the problem. This implies that the number of positive n such that $\sqrt{n^2 - 24}$ is an integer is equal to the number of negative n with the same property. We assume now that $n > 0$.

If k is a nonnegative integer such that $\sqrt{n^2 - 24} = k$ we get that $(n+k)(n-k) = 24$. Since $n+k$ and $n-k$ have the same parity (both even or both odd) and since $n+k > n-k$ we get that the only possibilities are: $\begin{cases} n+k = 12 \\ n-k = 2 \end{cases}$ and $\begin{cases} n+k = 6 \\ n-k = 4 \end{cases}$. This implies $n = 7$ and $n = 5$. Therefore we have 4 solutions $n = \pm 7$ and $n = \pm 5$.

31. $*$ Find how many integers k, $1 \le k \le 50$, have the property that

$$1 + 2 + \cdots + 49 + 50 - k$$

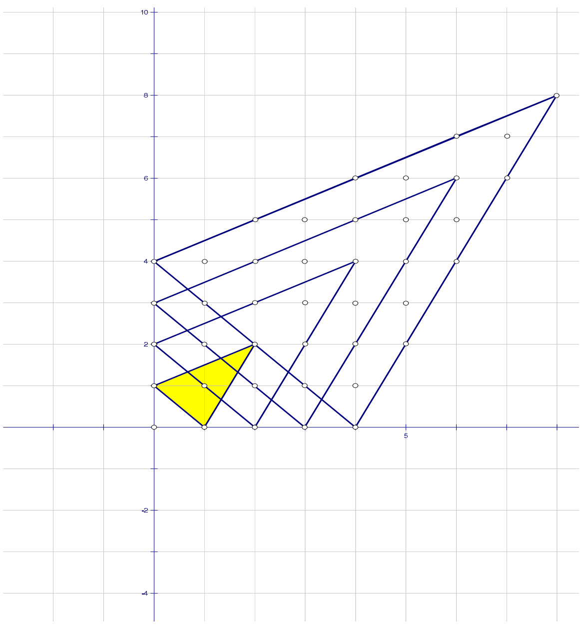
is a perfect square.

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Answer: B

We have that $1 + 2 + \cdots + 49 + 50 - k = \frac{50 \cdot 51}{2} - k = 1275 - k$. Since $35^2 = 1225$ and $36^2 = 1296$ and since $1 \le k \le 50$ we get that the only value that works is $k = 50$. Therefore there is only one integer k with the desired property.

32. In the adjacent figure we have a triangle T (shaded) with vertices $(1,0), (0,1)$, and $(2,2)$, and three of its dilations: $2T, 3T$, and $4T$. One can count the lattice points (integer coordinates) inside of the triangles. We denote by $I(n)$ the number of points of the lattice inside the dilation nT. For instance, $I(1) = 1$, $I(2) = 4$, and $I(3) = 10$. Find $I(9)$.



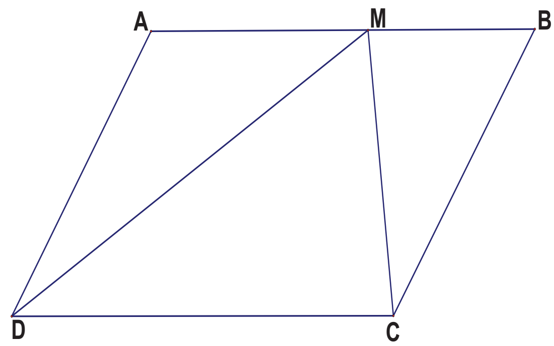
(A) 105 (B) 106 (C) 107

(D) 108 (E) 109

Answer: E

Note that we have the recursive formula $I(n+1) - I(n) = 3n$, for all integers $n \geq 1$. This implies that $I(9) = 109$.

33. Let $ABCD$ be a parallelogram and let M be a point on the segment $\overline{AB}$ such that the area of $\Delta MBC = 8 \text{ cm}^2$ and the area of $\Delta MDC = 20 \text{ cm}^2$. Find the area of the triangle ΔMAD.



(A) 4 cm^2 (B) 6 cm^2 (C) 8 cm^2

(D) 10 cm^2 (E) 12 cm^2

Answer: E

Denote by h the length of the perpendicular from A onto $\overline{BC}$. Then we have that $\dfrac{BM \cdot h}{2} = 8$ and that $\dfrac{DC \cdot h}{2} = 20$. Since $DC = BM + MA$ we get that the

$$\text{area of } \Delta MAD = \frac{MA \cdot h}{2} = \frac{(DC - BM) \cdot h}{2} = \frac{DC \cdot h}{2} - \frac{BM \cdot h}{2} = 20 - 8 = 12 \text{ cm}^2.$$

34. $*$ It is known that every positive integer can be written as the sum of non-consecutive Fibonacci numbers F_n in a unique way. Taking into consideration this writing for 2014, let

$$2014 = F_{n_1} + F_{n_2} + \cdots + F_{n_k}.$$

Find the number k.

(The Fibonacci numbers F_n are defined as follows: $F_1 = 1, F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$, for $n \geq 2$.)

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Answer: E

The largest Fibonacci number smaller or equal that 2014 is 1597 (that is, F_{17}). The next step is to take the largest Fibonacci number smaller or equal that 417 (2014-1597). That number is 377 (that is, F_{14}). Repeating this process we get $2014 = 1597 + 377 + 34 + 5 + 1$, so $k = 5$.

35. Find the product of the solutions of the equation $(4+\sqrt{15})^x + (4-\sqrt{15})^x = 62$.

(A) -25 (B) -16 (C) -9 (D) -4 (E) -1

Answer: D

If we denote $u = (4+\sqrt{15})^x$ then we have $\frac{1}{u} = (4-\sqrt{15})^x$. The equation $u + \frac{1}{u} = 62$ is equivalent to the quadratic equation $u^2 - 62u + 1 = 0$ and the quadratic formula gives us solutions $u_{1,2} = 31 \pm 8\sqrt{15}$. This implies that

$$x_1 = \log_{4+\sqrt{15}} u_1 = \frac{\ln(31+8\sqrt{15})}{\ln(4+\sqrt{15})} = \frac{\ln(4+\sqrt{15})^2}{\ln(4+\sqrt{15})} = 2.$$

$$x_2 = \log_{4+\sqrt{15}} u_2 = \frac{\ln(31-8\sqrt{15})}{\ln(4+\sqrt{15})} = -\frac{\ln(4+\sqrt{15})^2}{\ln(4+\sqrt{15})} = -2.$$

Therefore the product of the solutions is -4.

36. $*$ Find how many positive integers n have the property that $\sqrt{(n!)^2 + 13}$ is an integer, where $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$.

(A) 0 (B) 1 (C) 2

(D) 3 (E) infinitely many

Answer: B

If $n \geq 5$ then $(n!)^2 + 13$ has the digit 3 in the ones' place, so it can't be a perfect square. Of the remaining values, $n = 1, 2, 3, 4$, only $n = 3$ makes $\sqrt{(n!)^2 + 13}$ an integer.

37. Find the smallest positive integer n such that we can express 2014 as

$$2014 = \pm 1 \pm 2 \pm \cdots \pm n.$$

for some choice of the signs $\pm$. (For example, $8 = -1 + 2 + 3 + 4$ and the smallest n such that 8 can be written as $\pm 1 \pm 2 \cdots \pm n$ for some choice of the signs $\pm$ is $n = 4$.)

(A) 60 (B) 61 (C) 62 (D) 63 (E) 64

Answer: D

Note that $1 + 2 + 3 + \cdots + 63 = \dfrac{63 \cdot 64}{2} = 2016$. Thus we have

$$2014 = -1 + 2 + 3 + 4 + \cdots + 63.$$

It is clear that 60, 61, and 62 are too small. Even by taking all signs + will result in a sum smaller than 2014.

38. $*$ The sum of the first n positive integers is equal to a three digit number which has all digits equal. Find the sum of the digits of n.

(A) 9 (B) 13 (C) 17 (D) 21 (E) 25

Answer: A

We have that $1 + 2 + 3 + \cdots + n = \overline{aaa}$, where $a \in \{1, 2, 3, \cdots, 9\}$. This implies that $\dfrac{n(n+1)}{2} = \overline{aaa}$, so we have the quadratic equation $n^2 + n - 2 \cdot \overline{aaa} = 0$. Using the quadratic formula we get $n_{1,2} = \dfrac{-1 \pm \sqrt{1 + 8 \cdot \overline{aaa}}}{2}$. Since n has to be a positive integer we need to find a such that $1 + 8 \cdot \overline{aaa}$ is a perfect square. This implies $a = 6$, so $n = 36$. Therefore the sum of the digits of n is 9.

39. Let a and b be two integers such that the polynomial $f = X^4 - 2X^3 + aX^2 + bX + 1$ has the root $\dfrac{3 - \sqrt{5}}{2}$. Find the sum $a + b$.

(A) -6 (B) -5 (C) -4 (D) -3 (E) -2

Answer: D

Since the polynomial f has rational coefficients and $\frac{3-\sqrt{5}}{2}$ is a root we also have that $\frac{3+\sqrt{5}}{2}$ is a root. Denote by x_1 and x_2 the other two roots. By Viete's formulas we get that

$$x_1 + x_2 + \frac{3+\sqrt{5}}{2} + \frac{3-\sqrt{5}}{2} = 2.$$

This implies that $x_1 + x_2 = -1$. In addition we have

$$x_1 \cdot x_2 \cdot \frac{3+\sqrt{5}}{2} \cdot \frac{3-\sqrt{5}}{2} = 1,$$

so we get that $x_1 \cdot x_2 = 1$. Therefore

$$f = \left(X - \frac{3-\sqrt{5}}{2}\right)\left(X - \frac{3+\sqrt{5}}{2}\right)(X^2 + X + 1) = (X^2 - 3X + 1)(X^2 + X + 1).$$

Note that $f(1) = a + b$ (initial expression of f) and also $f(1) = -3$ (by the explicit formula found above). This implies $a + b = -3$.

40. $*$ Find the number of ordered triples of positive integers (x, y, z) with $x < y < z \leq 2014$ and $x^2 + y^2 + z^2 = xy + yz + zx + 3$.

(A) 2011 (B) 2012 (C) 2013 (D) 2014 (E) 2015

Answer: B

The equation can be written in the equivalent form

$$(x-y)^2 + (y-z)^2 + (z-x)^2 = 6.$$

Since $x < y < z$ we get that $y = x + 1$ and $z = x + 2$, so there are 2012 ordered triples of positive integers (x, y, z) such that $x < y < z \leq 2014$.

41. $*$ The number 2^{1230} is written after the number 3^{3450} to form a new number. How many digits does the new number have?

(A) 2014 (B) 2015 (C) 2016 (D) 2017 (E) 2018

Answer: E

Note that $\log 2^{1230} = 1230 \log 2 \approx 370.26$, so the number 2^{1230} has 371 digits. Similarly we have $\log 3^{3450} = 3450 \log 3 \approx 1646.07$, so 3^{3450} has 1647 digits. Therefore the number obtained by concatenating these two has $371 + 1647 = 2018$ digits.

42. Let a be a positive integer such that the graph of the quadratic function $y = ax^2+bx+c$ passes through the points $(-1, 4)$ and $(2, 1)$ and has two intersections with the x-axis. What is the maximum value of $b + c$?

(A) -1 (B) -2 (C) -3 (D) -4 (E) -5

Answer: D

We have that $\begin{cases} f(-1) = a - b + c = 4 \\ f(2) = 4a + 2b + c = 1. \end{cases}$ Subtracting these two equations we get $b = -1 - a$. We also obtain $c = 3 - 2a$, so we have that $b + c = 2 - 3a$. Because the equation has two intersections with the x-axis we have that $b^2 - 4ac > 0$. Considering the expressions of b and c in terms of a we get that $b^2 - 4ac = 9a^2 - 10a + 1 > 0$, or equivalently $(a - 1)(9a - 1) > 0$. This implies that $a \neq 1$ and since $a > 0$ we get that the smallest possible value for a is 2. Therefore $b + c = 2 - 3a \leq -4$.

43. Consider the number

$$M = (3^{2^0} + 1)(3^{2^1} + 1)(3^{2^2} + 1) \cdots (3^{2^{20}} + 1).$$

Which are the last two digits (base 10) of M?

(A) 20 (B) 22 (C) 24 (D) 26 (E) 28

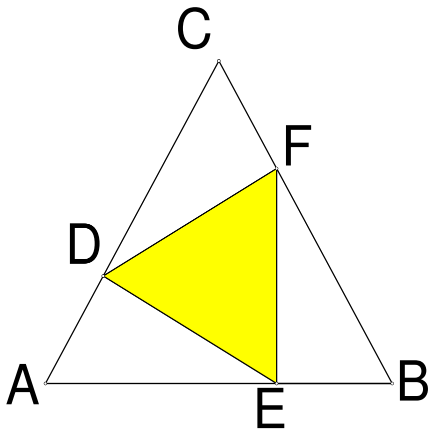
Answer: A

Note that $2M = (3^{2^0} - 1)M = (3^{2^1} - 1)(3^{2^1} + 1) \cdots (3^{2^{20}} + 1) = \cdots = 3^{2^{21}} - 1$. This is equivalent to $2M = 9^{2^{20}} - 1 = 9^{1048576} - 1 = (10 - 1)^{1048576} - 1$. Using the binomial formula, dividing by 2, and considering that we are looking for the last two digits of the number M it is enough to look at the terms $\frac{1}{2} \cdot \binom{1048576}{2} 10^2 - \frac{1}{2} \cdot \binom{1048576}{1} 10$. Since the last two digits of the term $\frac{1}{2} \cdot \binom{1048576}{2} 10^2$ are two zeroes it all comes down to the term $\frac{1}{2} \cdot \binom{1048576}{1} 10 = 5242880$. Therefore the last two digits of M are obtained by subtracting 80 from $\ldots 00$, so we get 20.

44. ∗ In the accompanying figure we have an equilateral triangle ABC and the points D, E, and F are on the sides $\overline{AC}$, $\overline{AB}$, and $\overline{BC}$ respectively. Knowing that

$$\frac{AC}{AD} = \frac{AB}{EB} = \frac{BC}{CF} = 3,$$

find the ratio of the area of ΔABC to the area of ΔDEF.



(A) 1 (B) 2 (C) 3

(D) 4 (E) 5

Answer: C

Denote by a the length of a side of the equilateral triangle ΔABC. Since

$$\frac{AC}{AD} = \frac{AB}{EB} = \frac{BC}{CF} = 3$$

we get that $AD = EB = CF = \dfrac{a}{3}$. This implies that $CD = AE = BF = \dfrac{2a}{3}$. We have that

$$\text{Area}_{\Delta ABC} = \text{Area}_{\Delta DCF} + \text{Area}_{\Delta ADE} + \text{Area}_{\Delta BFE} + \text{Area}_{\Delta DEF}.$$

This implies that

$$\text{Area}_{\Delta DEF} = \frac{a^2\sqrt{3}}{4} - 3 \cdot \frac{\frac{2a}{3} \cdot \frac{a}{3} \cdot \frac{\sqrt{3}}{2}}{2} = \frac{a^2\sqrt{3}}{12}.$$

Therefore

$$\frac{\text{Area}_{\Delta ABC}}{\text{Area}_{\Delta DEF}} = \frac{\frac{a^2\sqrt{3}}{4}}{\frac{a^2\sqrt{3}}{12}} = 3.$$

45. ∗ Find the positive integer n such that

$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \cdots + \frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left[\frac{1}{2} - \frac{1}{101 \cdot 102}\right].$$

(A) 98 (B) 99 (C) 100 (D) 101 (E) 102

Answer: C

We have that

$$\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \sum_{k=1}^{n} \left[\frac{1}{k(k+1)} - \frac{1}{k(k+2)}\right] = \sum_{k=1}^{n} \left[\frac{1}{k} - \frac{1}{k+1} - \frac{1}{2} \cdot \left(\frac{1}{k} - \frac{1}{k+2}\right)\right] =$$

$$= 1 - \frac{1}{n+1} - \frac{1}{2} \cdot \left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)} =$$
$$= \frac{1}{2} \cdot \left[\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right] = \frac{1}{2}\left[\frac{1}{2} - \frac{1}{101 \cdot 102}\right].$$

This implies $n = 100$.

46. Let x, y and z be three different real numbers such that

$$x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x}.$$

Which is the value of $x^2y^2z^2$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Answer: A

Note that $xyz \neq 0$. Let $k = x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x}$. Then we have $\begin{cases} xy + 1 = ky \\ yz + 1 = kz \\ zx + 1 = kx. \end{cases}$

This implies that $\begin{cases} xyz + z = kyz \\ xyz + x = kxz \\ xyz + y = kxy. \end{cases}$ These two systems imply that we have

$$\begin{cases} xyz = kyz - z = k(kz-1) - z = z(k^2-1) - k \\ xyz = kxz - x = k(kx-1) - x = x(k^2-1) - k \\ xyz = kxy - y = k(ky-1) - y = y(k^2-1) - k. \end{cases}$$

Since x, y, z are different real numbers we get that $k^2 = 1$, so $x^2y^2z^2 = 1$. For an example of such numbers, take $x = 2$, $y = -1$, and $z = \frac{1}{2}$.

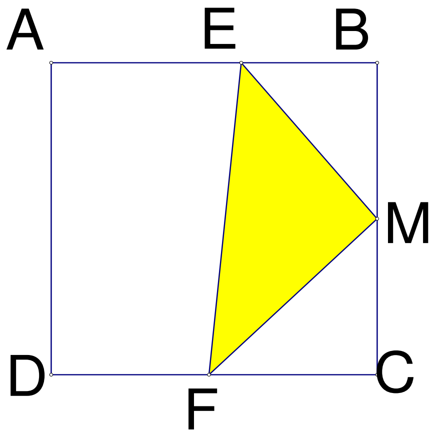
47. $*$ In a right triangle the legs a and b, measured in a certain unit, are positive integers such that $a > b$. Knowing that the hypothenuse, measured with the same unit, is $c = 2014$, what is $a - b$?

(A) 645 (B) 646 (C) 647 (D) 648 (E) 649

Answer: B

We have that a and b are integers such that $a > b$ and $b^2 + a^2 = 2014^2 = 2^2 \cdot 19^2 \cdot 53^2$. Since 19 is congruent to 3 modulo 4 we get that $b = 19b_1$ and $a = 19a_1$. This results in the equation $b_1^2 + a_1^2 = 2^2 \cdot 53^2$ or the equivalent equation, in Gaussian integers (in $\mathbb{Z}[i]$), $(b_1 + a_1 i)(b_1 - a_1 i) = (1+i)^2(1-i)^2(7+2i)^2(7-2i)^2$. This implies that one solution is $b_1 + a_1 i = (1+i)^2(7-2i)^2 = 2i(45-28i) = 56+90i$, so $b_1 = 56$ and $a_1 = 90$. The other possibilities, corresponding to a different choice of the factors, will be permutations of ± 56 and ± 90, with only one solution satisfying $a_1 > b_1$. Finally, we get $a - b = 19(a_1 - b_1) = 19 \cdot (90-56) = 646$.

48. $*$ In the accompanying figure we have a square $ABCD$ and M is the midpoint of the side $\overline{BC}$. If points E and F are chosen at random with uniform distribution on the sides $\overline{AB}$ respectively $\overline{CD}$, what is the probability that the angle $\angle EMF$ is acute?



(A) $\dfrac{1-\ln 2}{2}$ (B) $\dfrac{2-\ln 3}{3}$ (C) $\dfrac{3-\ln 4}{4}$

(D) $\dfrac{4-\ln 5}{5}$ (E) $\dfrac{5-\ln 6}{6}$

Answer: C

Without loosing the generality we may assume that $ABCD$ is a square with side 1. Suppose that the angle $\angle EMF = 90°$. Then triangles ΔEBM and ΔMCF are similar because they both have a $90°$ angle and $\angle BME + \angle CMF = 90°$. This implies that $\dfrac{EB}{\frac{1}{2}} = \dfrac{\frac{1}{2}}{FC}$, so we get that $FC = \dfrac{1}{4EB}$. Now, when the angle EMF is acute we have that $\dfrac{1}{4EB} \leq FC \leq 1$. Note that not every position of E on the side AB will determine an acute angle. For that to happen we need to have $BE \geq \dfrac{1}{4}$. If we denote the length of EB by x and that of FC by y then we have that the probability that the angle $\angle EMF$ is acute is the ratio between the measure (area) of the set

$$\{(x,y) | \frac{1}{4} \leq x \leq 1; \frac{1}{4x} \leq y \leq 1\}$$

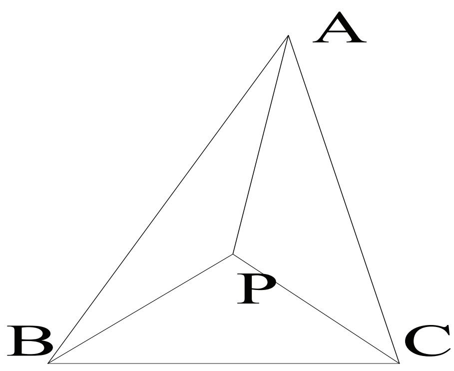
and the measure of $[0,1] \times [0,1]$. This implies that the probability is equal to

$$\int_{1/4}^{1} \left(1 - \frac{1}{4x}\right) dx = \left[x - \frac{1}{4}\ln x\right]_{1/4}^{1} = \frac{3-\ln 4}{4}.$$

49. In triangle ΔABC the angle $\angle ABC = 60°$. Let P be an interior point such that $\angle APB = \angle BPC = \angle CPA, PA = 8$ cm, and $PC = 6$ cm. What is the length of PB?

(A) $\sqrt{3}$ cm (B) $2\sqrt{3}$ cm (C) $3\sqrt{3}$ cm

(D) $4\sqrt{3}$ cm (E) $5\sqrt{3}$ cm



Answer: D

In triangle ΔPBC we have that $\angle PCB + \angle PBC = 60°$. Since $\angle ABP + \angle PBC = 60°$ we obtain that $\angle PCB = \angle ABP$. Since we also have $\angle APB = \angle BPC = 120°$ we obtain that triangles ΔAPB and ΔBPC are similar. Therefore we have

$$\frac{PA}{PB} = \frac{PB}{PC}.$$

This implies that $PB^2 = PA \cdot PC$, so $PB = 4\sqrt{3}$.

50. Let f be a positive continuous function defined on the real numbers which satisfies

$$f(x) = 3 + \int_0^x \frac{tf(t)}{1+t^2} dt,$$

for all real numbers x. What is $f(4/3)$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Answer: E

Note that $f(0) = 3$. Using the *Fundamental Theorem of Calculus* we get that

$$f'(x) = \frac{xf(x)}{1+x^2}, \text{ so } \frac{f'(x)}{f(x)} = \frac{1}{1+x^2} \text{ for all } x, \text{ since } f \text{ is positive.}$$

This implies that $\ln f(x) = \frac{1}{2}\ln(1+x^2) + C$. Since $f(0) = 3$ we get that $C = \ln 3$ and thus we have $\ln f(x) = \ln 3\sqrt{1+x^2}$. Therefore $f(x) = 3\sqrt{1+x^2}$ and $f\left(\frac{4}{3}\right) = 5$.