Solution to the First Annual Columbus State Pre-Calculus Tournament

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Precalculus Problems

1. *Find the remainder of the division* $(x^4 - 4x^3 + 6x^2 - 3x + 1) \div (x^2 - 2x + 1)$

   (A) $2x - 1$ (B) x (C) $3x - 2$ (D) $2 - x$ (E) $x - 1$

Solution: We observe first that $(x-1)^2 = x^2 - 2x + 1$ and one can check that $(x-1)^4 = x^4 - 4x^3 + 6x^2 - 4x + 1$ (or by looking at the Pascal's triangle). Hence,

$$x^4 - 4x^3 + 6x^2 - 3x + 1 = (x-1)^2(x-1)^2 + x,$$

which gives the answer B. ■

2. *For some positive numbers a and b we have the identity*

$$\frac{\sin 9x}{\cos 3x} + \frac{\cos 9x}{\sin 3x} = a \cot bx, \;\; x \in (0, \frac{\pi}{12}).$$

   What is $2a + b$?

   (A) 9 (B) 10 (C) 11 (D) 12 (E) 13

Solution: If we bring to the same denominator, the two fractions, and use the addition formula for *cosine* we get

$$E := \frac{\sin 9x}{\cos 3x} + \frac{\cos 9x}{\sin 3x} = \frac{\cos 9x \cos 3x + \sin 9x \sin 3x}{\sin 3x \cos 3x} = \frac{\cos(9x - 3x)}{\sin 3x \cos 3x}.$$

Using the double angle formula, $\sin(2\alpha) = 2 \sin \alpha \cos \alpha$, we can continue

$$E = \frac{\cos 6x}{\sin 3x \cos 3x} = 2\frac{\cos 6x}{\sin 6x} = 2 \cot 6x,$$

which gives $a = 2$ and $b = 6$. Then we obtain $2a + b = 10$ and so B is the correct answer. ■

3. *If θ is an angle in the third quadrant and $\tan\theta = \frac{28}{45}$, what is the value of $28\csc\theta$?*

(A) -63 (B) -45 (C) -47 (D) -50 (E) -53

Solution: The numbers 28, 45 and 53 form a Pythagorean triple ($53^2 - 45^2 = (53 - 45)(53 + 45) = 8(98) = 16(49) = 4^2(7^2) = 28^2$). In the third quadrant $\csc\theta = \frac{r}{y} = -\frac{53}{28}$ and so $28\csc\theta = -53$ which gives the answer E. ■

4. [$*^5$] *The cubic equation $2x^3 + 3x^2 + 5x + 2 = 0$ has two solutions, x_1 and x_2, which are not real numbers (pure complex). Find $x_1 + x_2$.*

(A) -1 (B) 1 (C) -2 (D) 2 (E) -3

Solution: We check to see if the given equation has any rational roots. The possible such roots are ± 1, ± 2 or $\pm(1/2)$. With a little luck one may find that $-1/2$ is indeed a zero, and so

$$2x^3 + 3x^2 + 5x + 2 = (2x + 1)(x^2 + x + 2)$$

which means the other two roots (which are indeed pure complex), by Viete's Relations, add up to -1 (Answer: A). ■

5. *Find the area of the triangle with sides $a = 9$, $b = 10$ and $c = 17$.*

(A) 20 (B) 22 (C) 24 (D) 36 (E) 18

Solution: Using Heron's formula we get $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{a+b+c}{2} = 18$, and so $s - a = 9$, $s - b = 8$, $s - c = 1$. Therefore, $A = \sqrt{18(9)(8)} = 3(6)2$ and so D is a correct answer. ■

6. [$*^4$] *If $t = \log_4 a = \log_6 b = \log_9(a - \frac{3}{2}b)$, what is $\frac{a}{b}$?*

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: From the first equality we see that $a = 4^t$, and then from the second equality, we have $b = 6^t$ and similarly $a - 3b/2 = 9^t$. Hence, $a(a - 3b/2) = 4^t(9^t) = 36^t = 6^{2t} = b^2$. This implies the equality $a^2 - 3ab/2 - b^2 = 0$ or if we denote by $x = a/b$ (note that $b \neq 0$), we get a quadratic equation in x: $x^2 - 3x/2 - 1 = 0$. We can solve this by using the quadratic formula or by completing the square: $(x - 3/4)^2 = 1 + 9/16 = 25/16$

which gives the only positive solution $x = 3/4 + 5/4 = 2$. Therefore $a/b = 2$ (Answer: B). ■

7. *The equation* $x^{\log_3 x} = \frac{x^3}{9}$ *has two solutions, say* x_1 *and* x_2*, with* $x_1 < x_2$*. What is* $x_2 - x_1$ *? Hint: Consider the logarithm base three of each side of the given equation.*

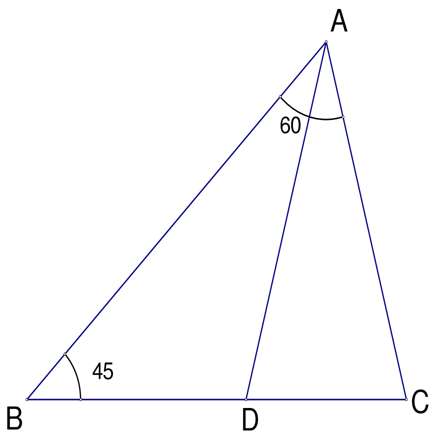
   (A) 2 (B) 4 (C) 6 (D) 8 (E) 10

Solution: Let us denote by $t = \log_3 x$ and (using the hint) observe that the given equation is equivalent to $\log_3(x^{\log_3 x}) = \log_3(\frac{x^3}{9})$ or $t^2 = 3t - 2$. This last quadratic equation can be solved simply by factorization: $(t-1)(t-2) = 0$. So, $x_1 = 3^1 = 3$ and $x_2 = 3^2 = 9$. So, C is the correct answer. ■

8. [$*^3$]*In the triangle* ABC *the angle* $\angle A$ *is* $60°$ *and the angle* $\angle B$ *is equal to* $45°$*. The angle bisector of the angle* $\angle A$ *intersects* $\overline{BC}$ *at* D*. Knowing that* $AD = 10$ *and that* $BC = m\sqrt{n}$*, where* m *and* n *are natural numbers with* n *not divisible by the square of a prime number, what is* $m + n$*?*

   (A) 19 (B) 15 (C) 11

   (D) 7 (E) 3



Solution: We observe that the angle $\angle ADC = 30° + 45° = 75°$ and the angle $\angle ACB = 180° - (60° + 45°) = 75°$. Hence, the triangle $\triangle ADC$ is an isosceles triangle, so $AD = AC = 10$. Using the Law of Sines in $\triangle ABC$, we get

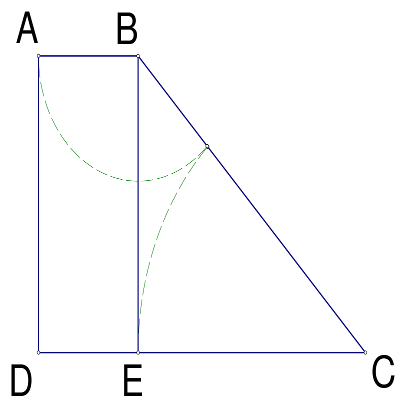
$$\frac{AC}{\sin 45°} = \frac{BC}{\sin 60°}$$

which implies $BC = 10\sqrt{3}/\sqrt{2} = 5\sqrt{6}$. This means $m = 5$ and $n = 6$. Hence the answer is C. ■

9. [$*^2$]*In the trapezoid $ABCD$, $\overline{AB}$ and $\overline{CD}$ are perpendicular to $\overline{AD}$. Knowing that $AD = 6 < BC = AB + CD$, what is $AB \cdot CD$?*

(A) 9 (B) 8 (C) 12

(D) 16 (E) 6

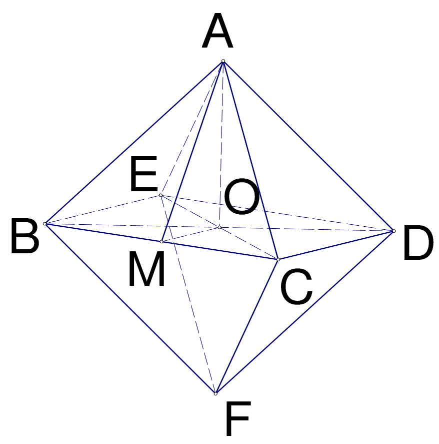


Solution: Without loss of generality, we may assume that $x := AB < y := DC$ (if these are equal, $ABCD$ is a rectangle, and this is not possible from the assumption $AD = 6 < BC$.) We observe that if $\overline{BE}$ is drawn perpendicular to $\overline{DC}$ (as in the figure associated) then $AB = DE = x$, $BE = 6$, $EC = y - x$, $BC = x + y$ and so, by the Pythagorean theorem in the triangle BEC we obtain $(x+y)^2 = 6^2 + (y-x)^2$ or $4xy = 36$. This implies $xy = 9$, so the answer is A. ■

10. [$*^1$] *In the accompanying figure we have a regular octahedron. We denote the dihedral angle between its faces by α (that is, $\alpha = 2(m\angle AMO)$). What is the value of $\cos\alpha$?*

(A) $-\frac{1}{2}$ (B) $-\frac{1}{3}$ (C) $-\frac{2}{3}$

(D) $-\frac{1}{4}$ (E) $-\frac{3}{4}$



Solution: As in the figure associated, we let M be the midpoint of $\overline{BC}$ and O the center of the square $BCDE$. In the triangle AMO we have $\cos\angle AMO = \cos\frac{\alpha}{2} = \frac{OM}{AM}$. Because of the symmetry, $OM = CD/2 = BC/2 = MC$. The side AM can be found with the Pythagorean theorem in the right triangle AMC: $AM^2 = AC^2 - MC^2 = 4MC^2 - MC^2 = 3MC^2$. Hence, $\cos\frac{\alpha}{2} = \frac{MC}{AM} = \frac{MC}{MC\sqrt{3}}$. Then, using the double angle formula we obtain

$$\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1 = \frac{2}{3} - 1 = -\frac{1}{3}.$$

Therefore, the answer in this case is B. ■