Thirty-Ninth Annual Columbus State Invitational Mathematics Tournament

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Solutions

1. There are 30 people in a room, 60% percent of whom are men. If no men enter or leave the room, how many women must enter the room so that 40% of the total number of people in the room are men?

   (A) 2 (B) 10 (C) 12 (D) 15 (E) 20

   Answer: D
   There are 18 men 60% of 30. Let x be the number of women to enter the room. Then percentage of men in the room is given by $\frac{18}{30+x} \cdot 100 = 40$. Solving for x yields $x = 15$.

2. What is the product of the roots of the equation $(x+4)(x+2)+(x+2)(x+6)=0$?

   (A) 2 (B) 10 (C) 24 (D) 48 (E) 96

   Answer: B
   Factoring $(x+4)(x+2)+(x+2)(x+6)=(x+2)(2x+10)=0$, the roots are $x=-2$ and $x=-5$, product is 10.

3. * If the domain for the function $f(x) = \frac{1}{x^2+2x+c}$ is $(-\infty, \infty)$, which of the following best describes all possible values of c ?

   (A) $c > 1$ (B) $c = 1$ (C) $c < 1$ (D) $c \leq 1$ (E) $c > 2$

   Answer: A
   Note $x^2+2x+c \neq 0$, thus the discriminant $b^2-4ac<0$, that is $4-4c<0$ and $c>1$.

4. * The quadratic polynomial $f(x)$ satisfies the equation $f(x)-f(x-2)=4x-2$ for all x. If s and t are the coefficients of x^2 and x, respectively, in $f(x)$, what is the value of $s+t$?

   (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

   Answer B
   $f(x)=sx^2+tx+c$ and $f(x-2)=s(x-2)^2+t(x-2)+c$. Simplifying and solving $sx^2+tx+c-(s(x-2)^2+t(x-2)+c)=4x-2$ yields $s=1$ and $t=1$ therefore $s+t=2$.

5. Max read on the website of Games and Stuff that a certain 100 dollar computer game was given a 32% discount; when Max arrived at the store, it was announced at the door that the same game had an additional 27% discount. When Max went to pay for it, he received a third discount of 17%. What single discount would have given the same cost as a series discounts of 32%, 27%, and 17%? Round your answer to two decimal places.

   (A) 71.68% (B) 76.00% (C) 58.80% (D) 41.20% (E) 28.40%

   Answer: C
   With first discount the cost is 68% of original price, with the second discount the cost is 73% of 68%, which is $\frac{73}{100} \cdot \frac{68}{100}$ of the original price. With the third discount, the cost will be 83% of $\frac{73}{100} \cdot \frac{68}{100}$, which yields $\frac{83}{100} \cdot \frac{73}{100} \cdot \frac{68}{100}$, ie the final cost is 41.20% of original cost. Single discount would be 58.80%.

6. Which of the following is equivalent to $\dfrac{2\sqrt{15}}{\sqrt{3}+\sqrt{5}+2\sqrt{2}}$?

   (A) $2\sqrt{3}+\sqrt{5}+2\sqrt{2}$ (B) $\sqrt{3}+2\sqrt{5}+\sqrt{2}$ (C) $-\sqrt{3}+\sqrt{5}+2\sqrt{2}$

   (D) $\sqrt{3}-\sqrt{5}+2\sqrt{2}$ (E) $\sqrt{3}+\sqrt{5}-2\sqrt{2}$

   Answer: E
   Rationalize the denominator, i.e multiply both the numerator and denominator by the conjugate $\sqrt{3}+\sqrt{5}-2\sqrt{2}$ which gives
   $$\frac{2\sqrt{15}}{\sqrt{3}+\sqrt{5}+2\sqrt{2}} \cdot \frac{\sqrt{3}+\sqrt{5}-2\sqrt{2}}{\sqrt{3}+\sqrt{5}-2\sqrt{2}} = \frac{2\sqrt{15}(\sqrt{3}+\sqrt{5}-2\sqrt{2})}{(\sqrt{3}+\sqrt{5})^2-8} = \sqrt{3}+\sqrt{5}-2\sqrt{2}.$$

7. You own a motel and have a pricing structure that encourages rentals of rooms in groups. One room rents for \$73, two for \$70.50 each, and in general the group rate per room is found by taking \$2.50 off the base of \$73 for each extra room rented. Find a formula for the function $R = R(n)$ that gives the total revenue for renting n rooms to a convention host.

   (A) $R(n) = 70.50 + 71(n+2)$ (B) $R(n) = 70.50 - 71(n-2)$

   (C) $R(n) = [73 - 2.50(n-1)]n$ (D) $R(n) = [73 - 2.50(n+1)]n$

   (E) $R(n) = [73 - 2.50(n-2.50)]n$

Answer: C

If n rooms are rented then the number of \$2.50 reductions is $(n-1)$. Thus the revenue of each room is $[73-2.50(n-1)]$ dollars. Total revenue is $R(n)=[73-2.50(n-1)]n$ dollars.

8. * If $\frac{1}{a}+\frac{1}{b}=\frac{1}{a+b}$, then find the value of $\frac{b}{a}+\frac{a}{b}$.

(A) -1 (B) 1 (C) 2 (D) -2 (E) 3

Answer: A

Multiplying both sides of the equation $\frac{1}{a}+\frac{1}{b}=\frac{1}{a+b}$ by $(a+b)$ gives $\frac{a+b}{a}+\frac{a+b}{b}=1$ and $1+\frac{b}{a}+\frac{a}{b}+1=1$. Thus $\frac{b}{a}+\frac{a}{b}=-1$.

9. * If the geometric mean[1] of two positive real numbers a and b $(a>b)$ is equal to 4 and their average is 5, then what is the value of x that satisfies the equation $a=b^x$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Answer: C

We have $0<b<a$, $\sqrt{ab}=4$ and $\frac{a+b}{2}=5$. So a and b satisfy $ab=14$ and $a+b=10$. Thus $a=8$ and $b=2$, so $8=2^x$ for $x=3$.

10. What is the sum of the real roots of $(x\sqrt{x})^x=x^{x\sqrt{x}}$?

(A) $\frac{18}{7}$ (B) $\frac{71}{4}$ (C) $\frac{9}{4}$ (D) $\frac{24}{19}$ (E) $\frac{13}{4}$

Answer: E

Writing in exponent form $x^{\frac{3}{2}x}=x^{x^{\frac{3}{2}}}$ for $x>0$ and $x\neq 1$. Solving for x in $\frac{3}{2}x=x^{\frac{3}{2}}$, we get $x=\frac{9}{4}$. Note $x=1$ is also a root. Sum of roots is $\frac{13}{4}$.

[1] The geometric mean of nonnegative numbers a and b is $\sqrt{ab}$.

11. Which of the following is the probability of getting exactly three sixes when you roll n fair dice?

(A) $\frac{n!}{3!}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^{n-3}$

(B) $\frac{n!}{3!(n-3)!}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^{n-3}$

(C) $\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^{n-3}$

(D) $\frac{n!}{3!(n-3)!}\left(\frac{1}{5}\right)^3\left(\frac{5}{6}\right)^{n}$

(E) $\left(\frac{1}{6}\right)^{n-3}\left(\frac{5}{6}\right)^{3}$

Answer:D
From $P(X=3)=\frac{n!}{3!(n-3)!}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^{n-3}$, simplify to get $\frac{n!}{3!(n-3)!}\left(\frac{1}{5}\right)^3\left(\frac{5}{6}\right)^{n}$.

12. If $3^{2-3y}=\pi$, compute the value of $\sqrt{3}^{(2+3y)}$.

(A) $\frac{18}{\pi}$ (B) $\frac{9}{\sqrt{\pi}}$ (C) π^3 (D) $\frac{9}{\pi}$ (E) 81π

Answer: B
From $3^{2-3y}=\pi$, we have $3^{3y}=\frac{9}{\pi}$. Thus $\sqrt{3}^{(2+3y)}=3\cdot(3^{3y})^{\frac{1}{2}}=\frac{9}{\sqrt{\pi}}$.

13. Three straight lines, l_1, l_2 and l_3 , have slopes $\frac{1}{2}$,$\frac{1}{3}$ and $\frac{1}{4}$, respectively. All three lines have the same y-intercept. If the sum of the x-intercepts of the three lines is 36, what is the y-intercept?

(A) $\frac{-13}{12}$ (B) $\frac{-12}{13}$ (C) -4 (D) 4 (E) 9

Answer: C
The equations of the lines l_1, l_2 and l_3 , are $y=\frac{1}{2}x+b$, $y=\frac{1}{3}x+b$ and $y=\frac{1}{4}x+b$ respectively.
The x-intercepts are $x=-2b$, $x=-3b$ and $x=-4b$. Summing them yields $-9b=36$, thus $b=-4$.

14. Evaluate

$$1+\frac{1}{\sqrt{2}+\sqrt{1}}+\frac{1}{\sqrt{3}+\sqrt{2}}+...+\frac{1}{\sqrt{2013}+\sqrt{2012}}.$$

(A) $1+\frac{1}{\sqrt{2013}}$ (B) $\sqrt{2013}$ (C) $\frac{2013}{\sqrt{2012}+\sqrt{2013}}$

(D) $2+\frac{1}{\sqrt{2013}}$ (E) $\frac{1}{\sqrt{2013}}$

Answer: B

Note that $\frac{1}{\sqrt{n}+\sqrt{n-1}}=\frac{\sqrt{n}-\sqrt{n-1}}{n-(n-1)}=\sqrt{n}-\sqrt{n-1}$. Therefore

$$1+\frac{1}{\sqrt{2}+\sqrt{1}}+\frac{1}{\sqrt{3}+\sqrt{2}}+...+\frac{1}{\sqrt{2013}+\sqrt{2012}}$$

$$=1+(\sqrt{2}-\sqrt{1})+(\sqrt{3}-\sqrt{2})+\cdots+(\sqrt{2012}-\sqrt{2011})+(\sqrt{2013}-\sqrt{2012}).$$

Which telescopes to $\sqrt{2013}$.

15. Three rugs have a combined area of 200 m^2 . By overlapping the rugs to cover a floor area of 140 m^2, the area which is covered by exactly two layers of rug is 24 m^2 . What area of floor is covered by three layers of rug?

(A) 12 m^2 (B) 18 m^2 (C) 24 m^2 (D) 36 m^2 (E) 42 m^2

Answer: B

Let x be the area covered by three layers, then we have one extra layer of 24 m^2 and two extra layers of x m^2. Thus $140+24+2x=200$, giving $x=18$.

16. In figure 1, the parabola has x-intercepts -1 and 4, and y-intercept 8. If the parabola passes through the point $(3, w)$, what is the value of w?

(A) 5 (B) 6 (C) 7

(D) 8 (E) 9

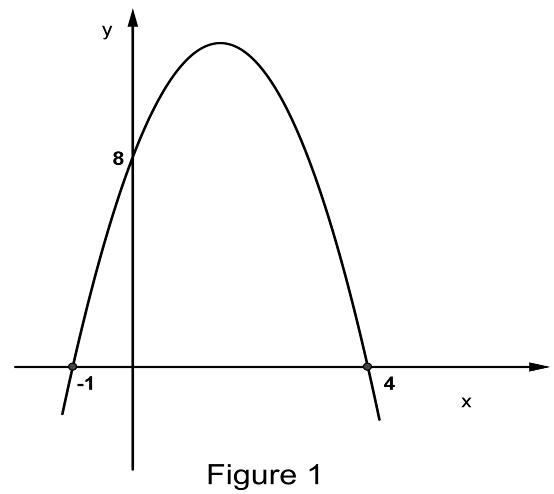

Figure 1

Answer: D
Use symmetry $f(0) = f(3)$, thus $w = 8$ (or find the quadratic function).

17. * What is the number of triples (x, y, z) such that when any of these numbers is added to the product of the other two, the result is 2 ?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Answer: B
We have
$x + yz = 2,$
$y + xz = 2,$
$z + xy = 2$
From the first two equations we obtain $x - y + yz - xz = 0$.
Factoring yield $(1 - z)(x - y) = 0$, thus $z = 1, x = y$. From $z = 1$, we have $(x, y, z) = (1, 1, 1)$. From $x = y$, we obtain $x + xz = 2$, $z + x^2 = 2$ from the first and third equations. Solving the system for x gives $(x - 1)^2(x + 2) = 0$, thus $x = 1$ and $x = -2$. Therefore $(x, y, z) = (-2, -2, -2)$.

18. Two six sided dice have each of their faces painted either blue or yellow. The first die has five blue faces and one yellow face. When the dice are rolled, the probability that the two top faces show the same color is $\frac{1}{2}$. How many yellow faces are there on the second die?

(A) 5 (B) 4 (C) 3 (D) 2 (E) 1

Answer: C

Once rolled the color of the second die must match the first with probability of $\frac{1}{2}$. This only happens if there are 3 blue and 3 yellow faces on the second die.

19. Write $\left(\frac{1}{\sqrt{2}}+\frac{\sqrt{2}}{2}i\right)^{2013}$ in the form $a+bi$.

(A) $\frac{-\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$ (B) $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$ (C) i

(D) -1 (E) $-i$

Answer: A

$$\left(\frac{1}{\sqrt{2}}+\frac{\sqrt{2}}{2}i\right)^{2013}=\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)^{2013}=\left(e^{i\frac{\pi}{4}}\right)^{2013}=e^{i(2012+1)\frac{\pi}{4}}=-e^{i\frac{\pi}{4}}=-\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right).$$

20. An increasing sequence is formed so that the difference between consecutive terms is a constant. If the first four terms of this sequences are x, y, $3x+y$ and $x+2y+2$, what is the value of $y-x$?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Answer:E

Let the common difference be d, so $y-x=d$, $3x+y-y=d$ and $x+2y+2-(3x+y)=d$. From the first and third equations we have $x=2$ and substituting in third equation we get $d=6$. Thus $y-x=6$.

21. * For $x<0$, simplify $\sqrt{(2x-|x|)^2}$.

(A) $-3x$ (B) $3x$ (C) 2x-$|x|$ (D) $|x|$ (E) x

Answer: A

For $x<0$, $(2x-|x|)=2x-(-x)=3x$, thus $(2x-|x|)<0$ and $\sqrt{(2x-|x|)^2}=|(2x-|x|)|=-(2x-|x|)=-3x$.

22. * Assume a and b are integers between 0 and 9, if **a679b** is the decimal representation of a number in base 10, such that it is divisible by 72, determine $a+b$.

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Answer D
Write **a679b** $= 10000 \cdot a + 6790 + b$ in $mod 8$. We get $6 + b \equiv 0 \ mod\ 8$, i.e $b = 2$. Expressing **a6792** $= 10000 \cdot a + 6790 + 2$ in $mod\ 9$, we obtain $a + 4 + 2 \equiv 0\ mod\ 9$, thus $a = 3$. Sum $a + b = 5$.

23. * What is the remainder when 2012^{2013} is divided by 7?

(A) 6 (B) 4 (C) 3 (D) 2 (E) 1

Answer:A

Write $2012 \equiv 3\ mod\ 7$. Since $3^6 \equiv 1\ mod\ 7$, we write $2012^{2013} \equiv (3^6)^{335} \cdot 3^3 \equiv 6\ mod\ 7$.

24. Three neon lights colored red, blue and green flash at different time intervals. The red light flashes after every 24 seconds, the blue light flashes after every 18 seconds and the green light after every 15 seconds. If all the three lights flash together at 8:00 am, how many times will all three lights flash together by 9:30 am?

(A) 12 (B) 15 (C) 18 (D) 21 (E) 24

Answer: B
Consider the least common multiple of $24, 18$ and 15 which is 360. The lights flash together every 6 minutes. In 90 minutes, all the three lights will flash together 15 times.

25. If $x = a^b$ and $y = \frac{1}{b} - \log_a \sqrt[b]{b}$,which expression is equivalent to x^y?

(A) $\frac{a}{b}$ (B) $a - \sqrt[b]{b}$ (C) $\frac{a}{\sqrt[b]{b}}$ (D) $a - b$ (E) $\frac{a}{b\sqrt[b]{b}}$

Answer: A

$$x^y = \left(a^b\right)^{\frac{1}{b} - \log_a \sqrt[b]{b}} = a^{1 - \log_a b} = \frac{a}{b}.$$

26. Find the exact value of the following product

$$\sin 1^\circ \cdot \sin 2^\circ \cdot ... \cdot \sin 45^\circ \cdot \sec 46^\circ \cdot ... \cdot \sec 89^\circ.$$

(A) 1 (B) $\frac{\sqrt{2}}{2}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{1}{2}$ (E) $\frac{1}{\sqrt{3}}$

Answer: B

From complementary angles

$$\sin 1^\circ \cdot \sin 2^\circ \cdot ... \cdot \sin 45^\circ \cdot \sec 46^\circ \cdot ... \cdot \sec 89^\circ$$
$$= \cos 89^\circ \cdot \cos 88^\circ \cdot ... \cdot \cos 46^\circ \cdot \sin 45^\circ \cdot \sec 46^\circ \cdot ... \cdot \sec 89^\circ.$$

The right side telescopes to $\sin 45^\circ = \frac{\sqrt{2}}{2}$

27. Given $x = \sqrt{c + \sqrt{c + \sqrt{c + ...}}}$, where x is a positive integer, which of the following is a possible values of c ?

(A) 9 (B) 10 (C) 16 (D) 18 (E) 20

Answer: E

Square both sides of the equation $x = \sqrt{c + \sqrt{c + \sqrt{c + ...}}}$, we get $x^2 = c + \sqrt{c + \sqrt{c + \sqrt{c + ...}}}$ i.e $x^2 = c + x$. From the resulting quadratic equation $x^2 - x - c = 0$, we need two numbers whose sum$= -1$ and product$= -c$. Thus c is a product of two consecutive numbers, namely 4 and 5.

28. In figure 2, the equation of the line containing $\overline{AD}$ is $y = \sqrt{3}(x - 1)$. $\overline{BD}$ bisects $\angle ADC$. If the coordinates of B are (p, q), what is the value of q?

(A) 6 (B) 6.5 (C) $\frac{10}{\sqrt{3}}$

(D) $\frac{12}{\sqrt{3}}$ (E) $\frac{13}{\sqrt{3}}$

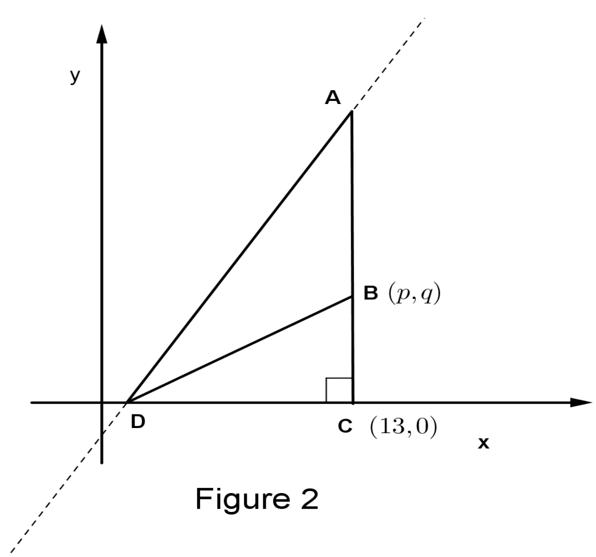

Figure 2

Answer: D

Slope of line segment$\overline{AD}$ is 3, thus $\tan\angle ADC = \sqrt{3}$ and so $\angle ADC = \frac{\pi}{3}$. We have $\angle BDC = \frac{\pi}{6}$ and $DC = 12$. Therefore $\tan\angle BDC = \frac{BC}{DC}$ yields $BC = \frac{12}{\sqrt{3}}$. Thus $q = \frac{12}{\sqrt{3}}$.

29. * What is the number of pairs of positive integers (s,t), with $s+t \leq 2013$, that satisfy the equation

$$\frac{s+t^{-1}}{s^{-1}+t} = 199?$$

(A) 5 (B) 7 (C) 8 (D) 10 (E) 15

Answer: D

Simplifying $\frac{s+t^{-1}}{s^{-1}+t} = 199$, we get $\frac{s}{t} = 199$. From $s+t \leq 2013$, we get $199t + t \leq 2013$ impling $t \leq \frac{2013}{200}$. Thus $1 \leq t \leq 10$ yields 10 such pairs (s,t).

30. * The numbers $a_1, a_2, ..., a_{61}$ are positive consecutive integers that sum to 2013. Which is the sum

$$a_1{}^2 + a_2{}^2 + ... + a_{61}{}^2 \quad ?$$

(A) 85338 (B) 85339 (C) 85340 (D) 85341 (E) 85342

Answer: B

From the sum $a_1, a_2, ..., a_{61} = 2013$, we get $a_1 = 3$ and $a_{61} = 63$. Since the numbers are consecutive $a_1{}^2 + a_2{}^2 + ... + a_{61}{}^2 = a_1{}^2 + (a_1+1)^2 + ... + (a_1+60)^2$. Thus $a_1{}^2 + a_2{}^2 + ... + a_{61}{}^2 = \frac{(63)(64)(127)}{6} - (2^2+1) = 85339$.

31. The quadratic equations $15x^2 - 19x + 6 = 0$ and $21x^2 - 17x + 2 = 0$ have a common solution. Which is the sum of the other solutions?

(A) $\frac{26}{35}$ (B) $\frac{24}{35}$ (C) $\frac{22}{35}$ (D) $\frac{28}{35}$ (E) $\frac{20}{35}$

Answer: A

Equating $15x^2-19x+6 = 21x^2-17x+2$ and solving for x yields the common solution $x=\frac{2}{3}$. Since $(x-\frac{2}{3})$ is a factor, solve for a in $(x-\frac{2}{3})(15x-a) = 15x^2-19x+6$ which yields $a=9$ which implies $x=\frac{3}{5}$ is the other solution of $15x^2-19x+6=0$. Similarly solve for b in $(x-\frac{2}{3})(21x-b) = 21x^2-17x+2$ to get $x=\frac{1}{7}$ as the other solution of $21x^2-17x+2=0$. The sum of other solutions is $\frac{26}{35}$.

32. * Let a_n equal the integer closest to $\sqrt{n}$. For example $a_1 = a_2$ since $\sqrt{1}=1$ and $\sqrt{2}\approx 1.41$ and $a_3=2$ since $a_3 \approx 1.73$.
What is the sum $\dfrac{1}{a_1}+\dfrac{1}{a_2}+\dfrac{1}{a_3}+\dfrac{1}{a_4}+\cdots+\dfrac{1}{a_{108}}+\dfrac{1}{a_{109}}+\dfrac{1}{a_{110}}$ equal to?

(A) 18 (B) 19 (C) 20 (D) 21 (E) 22

Answer: C

Trying a few terms in the sequence; $a_n=1$ for 2 values of n, $a_n=2$ for 4 values of n, $a_n=3$ for 6 values of n,...$a_n=k$ for $2k$ values of n. Since $\sqrt{110}\approx 10$ and $\sqrt{111}\approx 11$ we have includded all $a_n \leq 10$, implying 20 values of n.

Thus $\dfrac{1}{a_1}+\dfrac{1}{a_2}+\dfrac{1}{a_3}+\dfrac{1}{a_4}+\cdots+\dfrac{1}{a_{108}}+\dfrac{1}{a_{109}}+\dfrac{1}{a_{110}}$

$= 2(1)+4\left(\dfrac{1}{2}\right)+6\left(\dfrac{1}{3}\right)+...+20\left(\dfrac{1}{10}\right)=20.$

33. In figure 3, $AB=BC=CD=DA=BD=6$ and $AE=CE=14$. What is the length of line segment corresponding to $\overline{DE}$?

(A) $4\sqrt{10}-3$ (B) 11 (C) $7\sqrt{3}-3$

(D) 10 (E) 13

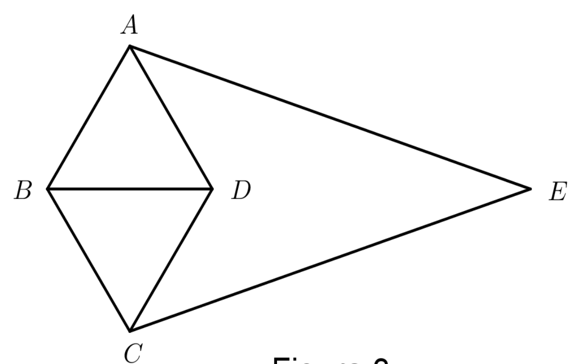

Figure 3

Answer: D

Let F be the intercestion of $\overline{AC}$ and $\overline{BD}$. Note $\angle ABD = 60°$ and $AF = 3\sqrt{3}$. Thus $FE = \sqrt{14^2 - (3\sqrt{3})^2} = 13$. Therefore $DE = FE - FD = 10$.

34. * Two points are chosen at random on a circle. What is the probability that the distance between them is more than the radius of the circle?

(A) $\frac{3}{4}$ (B) $\frac{2}{3}$ (C) $\frac{1}{2}$ (D) $\frac{1}{4}$ (E) $\frac{1}{3}$

Answer: B
Measure one radius from a point A on the circle, swing an arc from A any point on arc BAC is less than one radius away from A. This is $\frac{1}{3}$ of the circle. Any point on the other part which is $\frac{2}{3}$ of the circumfrence is greater than one radius away from point A. Thus Probability is $\frac{2}{3}$.

35. * The equation $2013 = x^2 + 41y^2$ has only two solution pairs (x, y) of positive integers. Considering these pairs, which is the largest value of $x + y$?

(A) 42 (B) 43 (C) 44 (D) 45 (E) 46

Answer: D
Consider $x < y$, then $x \leq 6$ and $y \geq 7$ yield the pair $(2, 7)$. For $x > y$, we have $y \leq 6$ and $x \geq 7$ yielding the pair $(43, 2)$. Thus the largest value of $x + y = 45$.

36. * Find the sum of all possible values of x satisfying the logarithmic equation

$$\log_{5x+9}\left(x^2 + 6x + 9\right) + \log_{x+3}\left(5x^2 + 24x + 27\right) = 4.$$

(A) $\frac{-5}{2}$ (B) $\frac{-3}{2}$ (C) 0 (D) 1 (E) $\frac{3}{2}$

Answer:A
Change of base yields $\frac{\log_{x+3}(x+3)^2}{\log_{x+3} 5x+9} + \log_{x+3}((5x+9)(x+3)) = 4$. Simplifying $\frac{2}{\log_{x+3} 5x+9} + \log_{x+3}((5x+9)) + 1 = 4$. Let $u = \log_{x+3}(5x+9)$ and solve the resulting quadratic equation. The solutions are $x = 0, x = -1$, and $x = \frac{-3}{2}$. Therefore sum of all possible values of x is $\frac{-5}{2}$.

37. *In figure 4, $ABCDEFGH$ is a cube of side-lengths equal to 12 meters. On the side AD there is an ant at the midpoint M of it. The ant travels on the faces of this cube to the point N located 3 meters away from G on FG. What is the shortest distance (in meters) that this ant can travel to arrive at N?

(A) $\sqrt{549}$ (B) $\sqrt{548}$ (C) $\sqrt{547}$

(D) $\sqrt{546}$ (E) $\sqrt{585}$

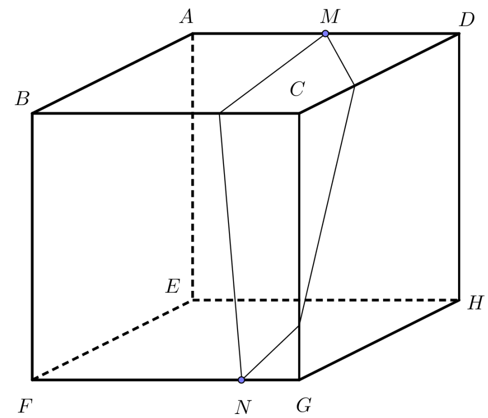

Figure 4

Answer:A

Openning up the cube into nets, the shortest distance from M to N is through face $CDHG$.

38. A line l is parallel to the line $y = \frac{5}{4}x + \frac{95}{4}$ and it intersects the x-axis and y-axis at points A and B, respectively. If l also passes through $(-1, -25)$, how many points with integer coordinates are there on the line segment $\overline{AB}$ (including the endpoints of $\overline{AB}$)?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Answer: B

The equation of line l is $y = \frac{5(x-19)}{4}$. To obtain integer coordinates $\frac{x-19}{4}$ must be an integer. Therefore $x = 3, x = 7, x = 11, x = 15$ and $x = 19$.

39. * Assume that two real numbers a and b satisfy $a^2 + ab + b^2 = 1$ and $t = ab - a^2 - b^2$. What is the range for the values of t ?

(A) $0 \leq t \leq \frac{2}{3}$ (B) $-\frac{1}{3} \leq t \leq 1$ (C) $-3 \leq t \leq -\frac{1}{3}$

(D) $-4 \leq t \leq -3$ (E) $1 \leq t \leq \frac{4}{3}$

Answer: C
We have $t + 1 = 2ab$ and $(a + b)^2 = \frac{t+3}{2} \geq 0$. So a and b are real roots of

$$x^2 \pm \sqrt{\frac{t+3}{2}}x + \frac{t+1}{2} = 0.$$

So $\frac{t+3}{2} - (2t + t) \geq 0$. Ths $-3 \leq t \leq -\frac{1}{3}$

40. * In figure 5 we have three circles of radii 3, 11 and 61 tangent to each other at points P, Q and R. The triangle formed by their centers area equal to $\sqrt{2013 \cdot a}$. What is the value of a?

(A) 71 (B) 72 (C) 73

(D) 74 (E) 75

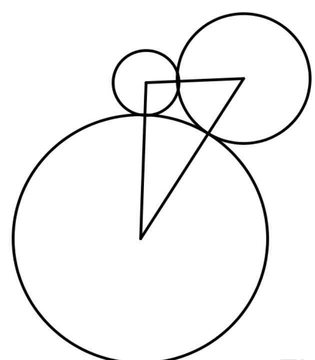

Figure 5

Answer: E
Using Hero's Formula $A = \sqrt{p(p-\alpha)(p-\beta)(p-\gamma)}$, where $p = \dfrac{\alpha+\beta+\gamma}{2}$ we have $\sqrt{75(3)(11)(61)} = \sqrt{75(2013)}$. Thus $a = 75$.

41. * Find the limit

$$\lim_{x\to 0} \frac{\sqrt[3]{5x+8}-2}{x}.$$

(A) $\frac{5}{12}$ (B) $\frac{7}{12}$ (C) $\frac{1}{12}$ (D) $\frac{1}{4}$ (E) $\frac{1}{2}$

Answer:A
Using L'Hopital's rule

$$\lim_{x\to 0} \frac{\sqrt[3]{5x+8}-2}{x} = \lim_{x\to 0} \frac{5}{3(5x+8)^{\frac{2}{3}}} = \frac{5}{12}.$$

42. * Two circles with radii a and b are tangent to each other as shown in figure 6. The ray $\overrightarrow{OA}$ contains the diameter of each circle, and the ray $\overrightarrow{OB}$ is tangent to each circle. Which of the following is equivalent to $\cos\theta$?

(A) $\frac{a+b}{b-a}$ (B) $\frac{2\sqrt{ab}}{a+b}$ (C) $\frac{2b-a}{\sqrt{ab}}$

(D) $\sqrt{\frac{a-b}{a+b}}$ (E) $\frac{a+b}{\sqrt{ab}}$

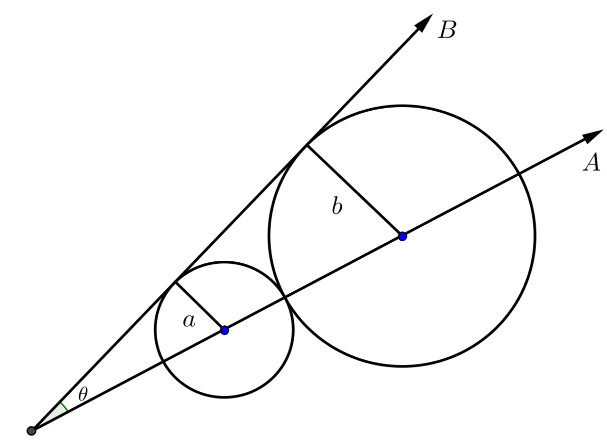

Figure 6

Answer: B
Note $\sin\theta = \frac{a}{h} = \frac{b}{h+a+b}$. Solving for $h = \frac{a(a+b)}{b-a}$ and $\cos\theta = \frac{2\sqrt{ab}}{a+b}$.

43. * The equation $9x^3 = a + \ln x$ has a unique positive solution in x if a is a certain positive real number. What is the value of $10e^{a-\frac{1}{3}}$?

(A) 34 (B) 33 (C) 32 (D) 31 (E) 30

Answer:E

The function $f(x) = 9x^3 - \ln x$ has a derivative equal to $f(x) = \frac{27x^3-1}{x}$ which shows that f has a global minimum at $x = \frac{1}{3}$ of $f(\frac{1}{3}) = \ln(\frac{1}{3}) = \frac{1}{3} + \ln 3$. Clearly a must be this minimum, and so $10e^{a-1/3} = 10e^{\ln 3} = 30$.

44. The square $ABCD$ in figure 7 has side lengths 4 meters. Point E is on $\overline{AC}$ with $AC = 4EC$. A circle centered at E is tangent to two sides of the square. $\overline{AG}$ is tangent to the circle at F. What is the length AF?

(A) $3\sqrt{2}$ (B) $\sqrt{17}$ (C) $\sqrt{21}$

(D) $4\sqrt{2} - 1$ (E) $4\sqrt{2}$

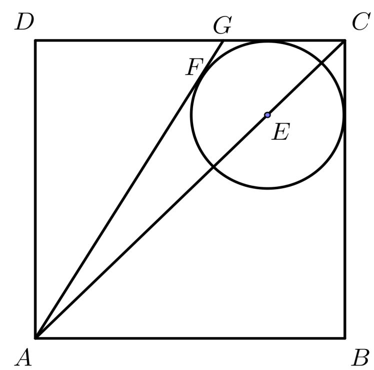

Figure 7

Answer: B

Diagonal AC has length $4\sqrt{2}$, $AE = 3\sqrt{2}$ and $FC = \sqrt{2}$. The radius $EF = 1$, hence $AF = \sqrt{17}$.

45. If every solution of the equation $(\cos x)^2 - (\cos x) - 1 = 0$ is a solution of the equation $a(\cos 2x)^2 + b(\cos 2x) - 1 = 0$, what is the value of $a + b$?

(A) -1 (B) -2 (C) -3 (D) -4 (E) -5

Answer: C

Use quadratic formula to obtain $\cos x = \dfrac{1-\sqrt{5}}{2}$. Plugging in $\cos 2x = 2 - \sqrt{5}$ into equation $a(\cos 2x)^2 + b(\cos 2x) - 1 = 0$, yields $a = 1$ and $b = -4$. Hence sum $a + b = -3$.

46. * In figure 8, the circle with center P has radius 3 and is tangent to both the positive x-axis and the positive y-axis, as shown. Also, the circle with center Q has radius 1 and is tangent to both the positive x-axis and the circle with center P. The line L is tangent to both circles. What is the y-intercept of L ?

(A) $8\sqrt{3}$ (B) $3+6\sqrt{3}$ (C) $10+3\sqrt{2}$

(D) $9+3\sqrt{3}$ (E) $10+2\sqrt{3}$

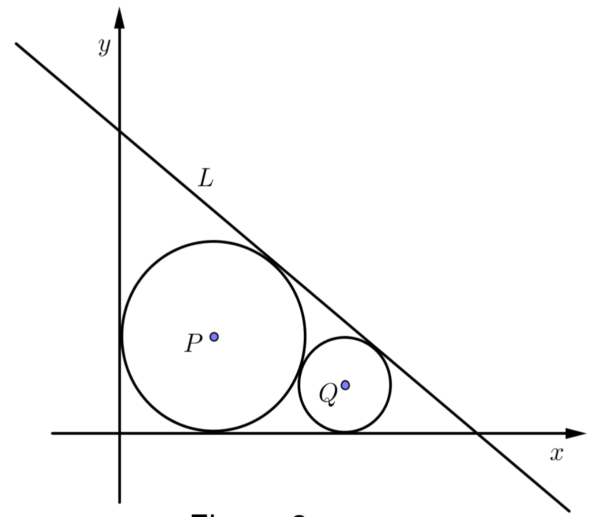

Figure 8

Answer: D

Let A and B indicate the y and x-intercepts of line L respectively. Let $QB = x$ and using similarity $\dfrac{x}{1} = \dfrac{4+x}{3}$, we get $x = 2$. We have $FB = \sqrt{3}$ and $\angle OBA = 60°$. By similarity we find $EF = 2\sqrt{3}$. From $\tan \angle OBA = \frac{OA}{OB}$, and we obtain $OA = 9 + 3\sqrt{3}$.

47. * The graph of the function g is the reflection of the graph of $f(x) = x + e^x$ defined for all real x about the line $y = x$. What is $g'(1)$ (i.e. $\dfrac{dg}{dx}(1)$) ?

(A) $\dfrac{1}{2}$ (B) $\dfrac{1}{3}$ (C) $\dfrac{1}{4}$ (D) $\dfrac{1}{5}$ (E) $\dfrac{1}{6}$

Answer: A

From the given information g is the inverse of f. So, $g(f(x)) = x$ implies
$g'(f(x))f'(x) = 1$.
Then for $x = 0$ we get $g'(f(0))f'(0) = 1$ or $g'(1) = \frac{1}{2}$.

48. * Knowing that there is exactly one pair (x, y) of two positive integers x and y satisfying the equation

$$\frac{x^2 + y^2}{33} = 2013,$$

what is $x + y$?

(A) 360 (B) 361 (C) 362 (D) 363 (E) 365

Answer: D

Decompose $2013 = 3(11)(61)$ and observe that $61 = 25 + 36 = 5^2 + 6^2$. Hence, we can take $x = 33(5)$ and $y = 33(6)$, which turns out to be the only solution of the proposed equation. Then $x + y = 33(11)$.

49. * How many ordered pairs (m, n) of positive integers are solutions of the equation $\dfrac{10}{m} + \dfrac{21}{n} = 1$?

(A) 11 (B) 12 (C) 14 (D) 16 (E) 18

Answer: D

$m = n = 31$ gives 1 pair (m, n). Consider $m < n$ and find values m such that $10 < m < 31$ for which $n = \dfrac{21m}{m - 10}$ is an positive integer. This yields 9 pairs (m, n). Consider $n < m$ and find values of n such that $21 < n < 31$ for which $m = \dfrac{10n}{n - 21}$ is a positive integer. This yields 6 pairs (m, n). Total number of ordered pairs (m, n) is 16.

50. * In figure 9, PQR is a right triangle with A and B on $\overline{PQ}$. Also, C is on $\overline{QR}$, and $\overline{BC}$ is parallel to $\overline{PR}$. If $AB = 2$, $PA = 3$, $PR = 4$, and the area of $\triangle ACR$ is 5, what is BQ ?

(A) 1 (B) 1.5 (C) $\sqrt{2}$

(D) 2 (E) $\frac{1}{3}$

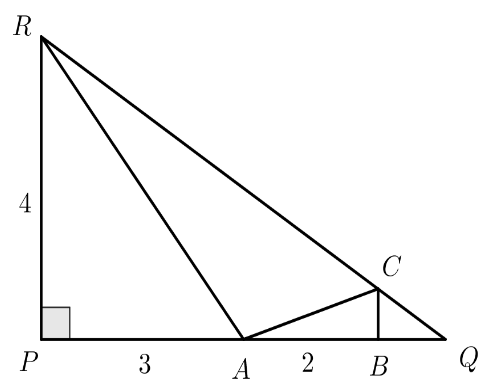

Figure 9

Answer: A

Let $BQ = x$ and $BC = h$. Area of $\triangle PQR = 10 + 2x$, area of $\triangle PRA = 6$, $\triangle RAC = 5$, $\triangle ACB = h$ and $\triangle CBQ = \frac{1}{2}xh$. Equating the areas we obtain the equation $xh + 2h - 4x + 2 = 0$, by similarity $\frac{5+x}{4} = \frac{x}{h}$ we obtain the equation $xh - 4x = -5h$.

Solving the two equations we obtain $h = \frac{2}{3}$ and $x = 1$.