Solutions

Math Tournament Ciphering

1. If $(x+1)(x-1)=8$, determine the numerical value of $(x^2+x)(x^2-x)$.

Solution:

Since $(x+1)(x-1)=8$, then $x^2-1=8$ and $x^2=9$.

Thus $(x^2+x)(x^2-x)=x^2(x+1)(x-1)=8x^2$

$$=72$$

Answer: 72

2. The function g(x) satisfies the equation $g(x)=g(x+1)+g(x-1)$ for all values of x. If $g(1)=1$ and $g(2)=3$, what is the value of $g(7)$?

Solution:

$$g(x+1)=g(x)-g(x-1)$$

So

$$g(3)=g(2)-g(1)=3-1=2$$

$$g(4)=g(3)-g(2)=2-3=-1$$

$$g(5)=g(4)-g(3)=-1-2=-3$$
$$g(6)=g(5)-g(4)=-3-(-1)=-2$$
$$g(7)=g(6)-g(5)=-2-(-3)=1.$$

Answer: $g(7)=1$

3. A carpenter wishes to construct a ladder with nine rungs whose lengths decrease uniformly from 24 inches at the base to 18 inches at the top. What is the length of the rung fourth from the top?

Solution:

$18 + 8d = 24$ Where d is the common difference. Solving for d yields;
$d = \frac{3}{4}$, So the length of each rung is increasing by $\frac{3}{4}$ of an inch.
The length of the fourth rung is given by $18 + 3d = 20\frac{1}{4}$ inches.

Answer: $20\frac{1}{4}$

4. A triangle has side-lengths equal to $8\sqrt{33}$, $15\sqrt{33}$ and$17\sqrt{33}$. What is its area?

Solution:
The triangle is a right triangle since$(8\sqrt{33})^2 + (15\sqrt{33})^2 = (17\sqrt{33})^2$. Hence area is given by $\frac{1}{2}(base)(height)$. Thus

Area= $\frac{1}{2}\left(8\sqrt{33}\right)\left(15\sqrt{33}\right) = (60)(33) = 1980$ sq. units.

Answer: 1980

5. Find the value of $\log_2 3 \cdot \log_3 4 \cdot \ldots \cdot \log_{2012} 2013 \cdot \log_{2013} 2$.

Solution**:**
Using change of base formula, rewrite in base e to obtain,
$\log_b a = \frac{lna}{lnb}$ for each of the log factors. We obtain

$$\frac{ln3}{ln2} \cdot \frac{ln4}{ln3} \cdot \frac{ln5}{ln4} \cdot \ldots \cdot \frac{ln2012}{ln2011} \cdot \frac{ln2013}{ln2012} \cdot \frac{ln2}{ln2013} = 1$$

Answer: 1

6. Compute the exact value of $(\sin 22.5^{\circ} - \cos 22.5^{\circ})^2$.

Solution:
Expansion gives $(\sin 22.5^{\circ})^2 - 2\sin 22.5^{\circ} \cdot \cos 22.5^{\circ} + (\cos 22.5^{\circ})^2$
Using trigonometric identities $\sin^2\theta + \cos^2\theta = 1$ and $\sin 2\theta = 2\sin\theta\cos\theta$, we obtain

$$(\sin 22.5^{\circ})^2 - 2\sin 22.5^{\circ} \cdot \cos 22.5^{\circ} + (\cos 22.5^{\circ})^2$$
$$= (\sin 22.5^{\circ})^2 + (\cos 22.5^{\circ})^2 - \sin 45^{\circ} = 1 - \frac{\sqrt{2}}{2}$$

Answer: $1 - \frac{\sqrt{2}}{2}$

7. Suppose the functions $f(x)$ and $g(x)$ satisfy the system of equations

$$f(x) + 3g(x) = x^2 + x + 6$$
$$2f(x) + 4g(x) = 2x^2 + 4$$

for all x. Find $g(x)$.

Solution:
The second equation has a common factor 2, so the system reduces to

$$f(x) + 3g(x) = x^2 + x + 6$$
$$f(x) + 2g(x) = x^2 + 2$$

Subtracting the second equation from the first yields $g(x) = x + 4$.

Answer: $g(x) = x + 4$

8. If $ac + ad + bc + bd = 102$ and $c + d = 6$, what is the value of $a + b + c + d$?

Solution: Factoring gives
$ac + ad + bc + bd = a(c + d) + b(c + d) = (c + d)(a + b) = 102.$
Since $c + d = 6$, then $a + b = 17$ and $a + b + c + d = 23$.

Answer: 23

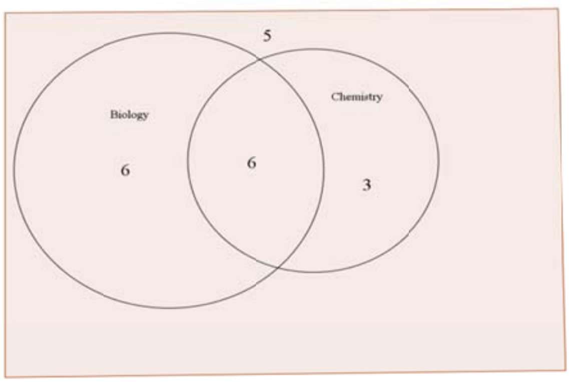
Solutions Ciphering Round 2

1. Two consecutive positive integers and satisfy the equation
What is the value of ?

Solution: Factoring yields . Factors of 6 are (2, 3) and (1, 6). Since and are consecutive and then and .

Answer:

2. In a group of 20 students taking Sciences, 6 of them are taking both Chemistry and Biology, 9 of them are taking Chemistry and 12 students are taking Biology.
What is the percentage of students taking neither Chemistry nor Biology?



Answer: 25%

3. If and where and are positive real numbers, what is the value of ?

Solution:

Thus and , giving and

Answer:

4. Given angles A and B such that,

$$\sin^2 A + \cos^2 B = \frac{3}{2}x$$
$$\cos^2 A + \sin^2 B = \frac{1}{2}\ x$$

Determine the value of x.

Solution:
Adding the two equations, and using the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$, we obtain

$$2x = 2, \qquad x = 1$$

Answer: $x = 1$

5. Evaluate $\frac{1}{8} + \log_2 2\sqrt{2\sqrt{2}}$.

Solution:
Using properties of logarithms
$\log_2 2\sqrt{2\sqrt{2}} = \log_2 2 + \frac{1}{2}(\log_2 2 + \frac{1}{2}\log_2 2)$.
Noting $\log_2 2 = 1$, $\frac{1}{8} + \log_2 2\sqrt{2\sqrt{2}} = \frac{1}{8} + \frac{7}{4} = \frac{15}{8}$

Answer: $\frac{15}{8}$

6. Let α be a solution of $x^2 + x + 1 = 0$. What is the value of
$\alpha^6 + \alpha^5 + \alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1$?

Solution: Factoring by grouping three terms from the left yields
$\alpha^4(\alpha^2 + \alpha + 1) + \alpha(\alpha^2 + \alpha + 1) + 1 = 1$. Since $\alpha^2 + \alpha + 1 = 0$.
Answer: 1

7. Rectangle ABCD is made up of six squares. The areas of two of the squares are shown. Find the perimeter of rectangle ABCD, in centimeters.

   Solution: The lengths of the squares with area $36\ cm^2$ and $25\ cm^2$ are $6\ cm$ and $5\ cm$ respectively. The smallest of the six squares has length $1cm$, the larges square has length $7cm$. The two other squares have equal lengths $4cm$, thus the rectangle has length $13cm$ and width $11\ cm$. Perimeter is $48\ cm$.

   Answer: $48\ cm$

8. If $S = 3^{2013} + 3^{-2012}$ and $T = 3^{2013} - 3^{-2012}$, what is the exact value of $S^2 - T^2$?

   Solution: $S^2 - T^2 = (S+T)(S-T) = (2 \cdot 3^{2013})(2.\,3^{-2012}) = 12$

   Answer: 12