MATH TOURNAMENT 2012 PROBLEMS – SOLUTIONS

1. Consider the experiment of throwing two 6–sided fair dice, where, the faces are numbered from 1 to 6. What is the probability of the event that the sum of the values of the two dice is 7?

(a) $\frac{1}{12}$ (b) $\frac{1}{9}$ (c) $\frac{1}{6}$ (d) $\frac{2}{9}$ (e) $\frac{1}{4}$

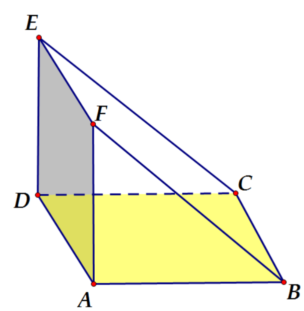
There are 36 possible outcomes of which 6 outcomes (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) with sum = 7. Therefore P = 6/36 = 1/6

2. Find the last digit of the sum $0! + 2! + 4! + \cdots + 2010! + 2012!$.

(a) 1 (b) 3 (c) 5 (d) 7 (e) 9

After 6!, all of them end with 0. It suffices therefore to just look at the first 3 terms whose sum is 1 + 2 + 24 = 27. Therefore the last digit must be 7.

3. Squares *ABCD* and *ADEF* are in perpendicular planes. If $AB = 4$, find *FC*.



(a) $2\sqrt{3}$ (b) $4\sqrt{2}$ (c) $3\sqrt{5}$ (d) $4\sqrt{3}$ (e) $2\sqrt{5}$

This problem is basically asking for the length of the main diagonal of a cube of side length 4. Pythagorean Theorem yields:

$FC^2 = FA^2 + AC^2 = FA^2 + AB^2 + BC^2 = 16 + 16 + 16 = 48$ which implies FC = $4\sqrt{3}$

4. The probability distribution of your winnings at a casino's card game is shown below.

X	\$0	\$5	\$10	\$25
$P(X)$	0.1	0.4	0.3	0.2

How much should you expect to win if you play the game once?

(a) \$5 (b) \$7.5 (c) \$10 (d) \$12.5 (e) \$15

$\sum XP(X) = 0 \cdot 0.1 + 5 \cdot 0.4 + 10 \cdot 0.3 + 25 \cdot 0.2 = 10\$$

5. Working alone, Al can wash a hotel restaurant's dishes in three hours. Jenny can do the same job in six hours. If they worked together, how long would it take them to complete this task?

(a) 9 hours (b) $\frac{9}{2}$ hours (c) 2 hours (d) $\frac{2}{9}$ hours (e) $\frac{1}{9}$ hours

$\frac{3 \cdot 6}{3+6} = 2$ hours

6. What is the sum of all positive integer divisors of 2012?

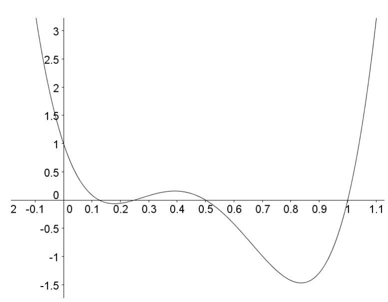
(a) 1034 (b) 1724 (c) 2347 (d) 3528 (e) 5213

$1 + 2 + 4 + 503 + 1006 + 2012 = 3528$

7. How many roots of the equation $1 - 15x + 70x^2 - 120x^3 + 64x^4 = 0$ lie in the interval $[0,1]$?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

The graph of the function $f(x) = 1 - 15x + 70x^2 - 120x^3 + 64x^4$ has 4 x-intercepts, all of which lie in the interval [0, 1]

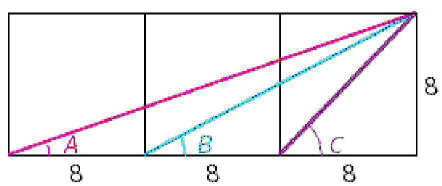


8. Polynomials P and Q satisfy the equation $P(x-2)=(x^2+1)\cdot Q(x-1)-x-1$ for all real numbers x. When $P(x)$ is divided by $(x-3)$, the remainder is 20. Determine the remainder when $Q(x)$ is divided by $(x-4)$.

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

The remainder when $P(x)$ is divided by $(x-3)$ is $P(3)=20$.
The remainder when $Q(x)$ is divided by $(x-4)$ is $Q(4)$.
Setting $x=5$ in the original equation, we get:
$P(5-2)=P(3)=(5^2+1)\cdot Q(5-1)-5-1$
Simplifying yields $20=26\cdot Q(4)-6$
Solving for $Q(4)$ we get $Q(4)=1$.

9. The following is a rectangle made of three congruent squares of side length 8 units. Determine $m\angle A+m\angle B+m\angle C$.



(a) 67.5° (b) 78.75° (c) 90° (d) 101.25° (e) 112.5°

Angle C is one of the base angles of a right isosceles triangle. Therefore $m\angle C=45°$.
To determine $m\angle A+m\angle B$, we use the tangent of the sum formula:

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}=\frac{\frac{8}{24}+\frac{8}{16}}{1-\frac{8}{24}\frac{8}{16}}=\frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\frac{1}{2}}=\frac{\frac{5}{6}}{\frac{5}{6}}=1$$, which implies that

$m\angle A+m\angle B=45°$ as well. Therefore $m\angle A+m\angle B+m\angle C=90°$

10. The rational function $f(x)=\dfrac{x^3-8}{x^2-4}$ is defined for all real numbers except for $x=\pm 2$. Given that it has a vertical asymptote $x=a$ and a slant asymptote $y=mx+b$, determine the sum $a+m+b$.

(a) -3 (b) -1 (c) 0 (d) 1 (e) 3

The rational function $f(x)=\frac{x^3-8}{x^2-4}$ has a vertical asymptote $x=-2$ and a slant asymptote $y=x$. Therefore, the sum $a+m+b=-2+1+0=-1$.

11. If $N=2^{6n+2}+4^{3n+2}+8^{2n+1}$, what is the smallest positive integer x such that the product xN is a perfect square for all positive integers n?

(a) 2 (b) 3 (c) 5 (d) 7 (e) 11

$N=2^{6n+2}+4^{3n+2}+8^{2n+1}=2^{6n+2}+2^{6n+4}+2^{6n+3}=2^{6n+2}\left(1+2^2+2^1\right)=7\cdot 2^{6n+2}=7\cdot\left(2^{3n+1}\right)^2$

Therefore, the smallest positive integer x such that the product xN is a perfect square for all positive integers n must be 7.

12. Let $A=\{1,3,4\}$ and $B=\{1,2,3,4,5,6,7\}$. How many subsets of B contain A?

(a) 16 (b) 32 (c) 48 (d) 96 (e) 112

$B-A=\{2,5,6,7\}\Rightarrow n(B-A)=4$

Therefore, the number of subsets of set $B-A$ is $2^4=16$.

Because $(B-A)\cup A=B$, if we add A to these 16 subsets, we conclude that the 16 subsets of B contain A.

13. Determine the value of the product $(\tan 1°)(\tan 2°)(\tan 3°)\ldots(\tan 88°)(\tan 89°)$.

(a) 0 (b) 1 (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$ (e) $\frac{\pi}{6}$

Using the fact that $\tan x=\cot(90-x)=\frac{1}{\tan(90-x)}$, we see that all terms cancel except the term in the middle, $\tan 45°=1$.

14. Determine the solution set of the equation $\log_3\left(9\cdot 3^{x+3}\right)=3|x|+1$.

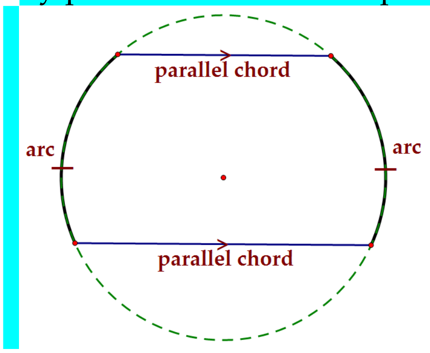
(a) $\{-1,2\}$ (b) $\{0,2\}$ (c) $\{0,3\}$ (d) $\{4,0\}$ (e) $\{2,3\}$

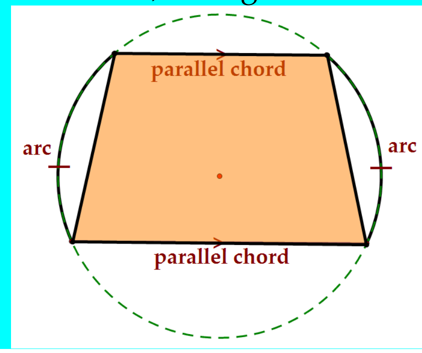
Converting to the exponential form, we obtain $(9\cdot 3^{x+3}) = 3^{3|x|+1} \Rightarrow 3^{x+5} = 3^{3|x|+1}$
$\Rightarrow x+5 = 3|x|+1 \Rightarrow x+4 = 3|x|$
Case 1: $x<0$: Solve $x+4=-3x \Rightarrow x=-1$
Case 1: $x\geq 0$: Solve $x+4=3x \Rightarrow x=2$

15. Quadrilateral *ABCD* is inscribed in a circle in such a way that exactly one pair of opposite sides intercept congruent arcs. Being as specific as possible, what type of quadrilateral is *ABCD*?

(a) Square (b) Rectangle (c) Kite (d) Trapezoid (e) Parallelogram

By parallel chords intercepting congruent arcs theorem, the figure must be a trapezoid.





16. Let $x=\sqrt{7+\sqrt{13}}-\sqrt{7-\sqrt{13}}$. Evaluate the expression $(x^3-2x-2)^{2012}$.

(a) 2^{8048} (b) 2^{2024} (c) 4^{1006} (d) 4^{503} (e) 8^{6036}

$x^3-2x-2 = x(x^2-2)-2$
$x^2 = 7+\sqrt{13}+7-\sqrt{13}-2\sqrt{49-13} = 14-2\sqrt{36} = 2$
Therefore $(x^2-2)=0 \Rightarrow x^3-2x-2=-2 \Rightarrow (x^3-2x-2)^{2012} = (-2)^{2012} = 2^{2012} = 4^{1006}$

17. In space, which of the following propositions is false?

(a) A line parallel to one of two parallel lines is also parallel to the other line.
(b) Three parallel lines may not be coplanar.
(c) A line intersecting one of two parallel lines must intersect the other line as well.
(d) Through a point not on a given plane, there is exactly one plane parallel to the given plane.
(e) Through two distinct points in a plane, there is exactly one plane perpendicular to that

plane.

All propositions are true except (c).

18. In the experiment of selecting 5 marbles without replacement from a bag containing 6 yellow and 4 red marbles, what is the probability of getting 3 yellow and 2 red marbles?

(a) $\frac{1}{3}$ (b) $\frac{2}{5}$ (c) $\frac{10}{21}$ (d) $\frac{1}{2}$ (e) $\frac{5}{9}$

$$\frac{\binom{6}{3}\binom{4}{2}}{\binom{10}{5}} = \frac{10}{21}$$

19. What is the product of all integers n such that $\sqrt{n^2 - 37}$ is an integer?

(a) -225 (b) -324 (c) -361 (d) -441 (e) -576

The expression $n^2 - 37$ must be a perfect square. Let $n^2 - 37 = k^2$, with k integer. Then $n^2 - k^2 = 37 \Rightarrow (n-k)(n+k) = 37$, which will give the following systems:

$$\begin{matrix} n-k=37 & n-k=-37 & n-k=1 & n-k=-1 \\ n+k=1 & n+k=-1 & n+k=37 & n+k=-37 \end{matrix}$$

which will happen only if $n = \pm 19$. Therefore the product is –361.

20. How many real solutions does the equation $\left(x^2 - 2\right)^{x^2+2x} = 1$ have?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

Case 1: $x^2 - 2 = 1 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$
Case 2: $x^2 + 2x = 0 \wedge x^2 - 2 \neq 0 \Rightarrow x = 0,-2$
Case 3: $x^2 - 2 = -1 \wedge x^2 + 2x$ is even. $\Rightarrow x^2 = 1 \wedge x^2 + 2x$ is even. $\Rightarrow x = \pm 1 \wedge x^2 + 2x = 3,-1$ is odd, namely false.
Therefore, there are only 4 real solutions.

21. Let $E_n = \sum_{k=0}^{n} \binom{n}{k}$ with n a natural number. Evaluate the difference $E_{n+1} - E_n$.

(a) $2^n - 1$ (b) 2^{n-1} (c) 2^n (d) 2^{n+1} (e) $2^n + 1$

$$E_{n+1} - E_n = \sum_{k=0}^{n+1} \binom{n+1}{k} - \sum_{k=0}^{n} \binom{n}{k} = 2^{n+1} - 2^n = 2^n$$

22. Evaluate the expression $\sum_{n=1}^{10} \sum_{m=2}^{8} (mn - 3n)$.

(a) -2012 (b) -1006 (c) 0 (d) 770 (e) 2012

$$\sum_{n=1}^{10} \sum_{m=2}^{8} (mn - 3n) = \sum_{n=1}^{10} \sum_{m=2}^{8} n(m-3) = \sum_{n=1}^{10} n \sum_{m=2}^{8} (m-3) = \frac{10 \cdot 11}{2} \cdot (-1+0+1+2+3+4+5) = 770$$

23. If a projectile is fired with velocity v at an angle θ, then its vertical displacement y as a function of its horizontal displacement x is modeled by the parabola $y = -\frac{5}{v^2 \cos^2 \theta} x^2 + (\tan \theta) x$. Find an expression for the range R (the maximum horizontal displacement) of this projectile.

(a) $\frac{v^2 \sin 2\theta}{10}$ (b) $\frac{v^2 \sin \theta}{5}$ (c) $\frac{v^2 \sin \theta \cos \theta}{10}$ (d) $\frac{v^2 \cos^2 \theta}{20}$ (e) $\frac{v^2 \sin^2 \theta}{20}$

Setting $y = 0$ and solving for x, we get:

$$x\left(-\frac{5}{v^2 \cos^2 \theta} x + \tan \theta\right) = 0 \Rightarrow x = 0, \frac{v^2 \cos^2 \theta \tan \theta}{5}$$

Simplifying, the range is $\frac{v^2 \cos^2 \theta \tan \theta}{5} = \frac{v^2 \cos^2 \theta \frac{\sin \theta}{\cos \theta}}{5} = \frac{v^2 \cos \theta \sin \theta}{5} = \frac{v^2 \sin 2\theta}{10}$

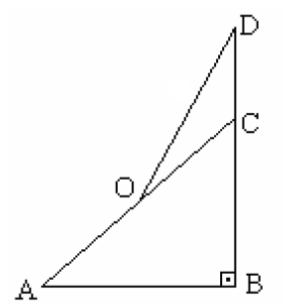
24. When a camera flash goes off, the batteries immediately begin to recharge the flash's capacitor, which stores electric charge given by $Q(t) = Q_0 \left(1 - e^{-t/4}\right)$. (The

maximum charge capacity is Q_0 and t is measured in seconds). How long (in seconds) does it take to recharge the capacitor to within 10% of its capacity?

(a) $-4\ln 1$ (b) $-4\ln\frac{9}{10}$ (c) $-4\ln 10$ (d) $-4\ln\frac{1}{10}$ (e) $-\ln\frac{4}{9}$

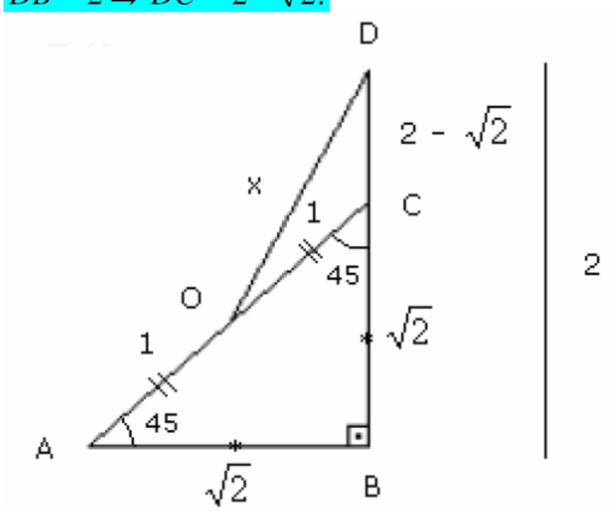
Solving the equation $0.9Q_0 = Q_0\left(1-e^{-t/4}\right)$ for t, we get $t = -4\ln\frac{1}{10}$

25. ABC is an isosceles right triangle. $\overline{BC}$ is extended such that B, C, and D are collinear, and O is the midpoint of $\overline{AC}$. If $BD = AC = 2$, find OD.



(a) $\sqrt{3-\sqrt{2}}$ (b) $\sqrt{4-2\sqrt{2}}$ (c) $\sqrt{5-\sqrt{3}}$ (d) $\sqrt{4-\sqrt{2}}$ (e) $\sqrt{5-2\sqrt{2}}$

Because ΔABC is a right isosceles triangle and $AC = 2$, the Pythagorean Theorem gives $AB = BC = \sqrt{2}$. Moreover, $m\angle BAC = m\angle BCA = 45°$.
$AO = OC = 1$.
$DB = 2 \Rightarrow DC = 2-\sqrt{2}$.



Applying Cosine Law for triangle ΔDOC yields:

$$x^2 = 1^2 + (2 - \sqrt{2})^2 - 2.1.(2 - \sqrt{2}).\cos 135$$

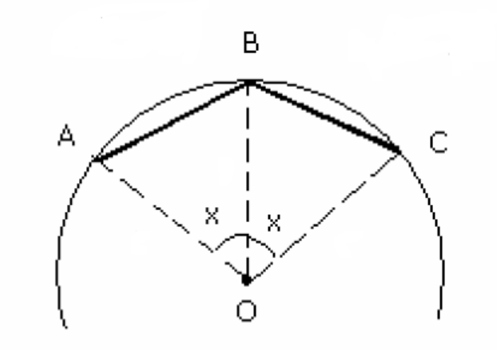
$$\Rightarrow \quad x^2 = 1 + 4 - 4\sqrt{2} + 2 - 2.(2 - \sqrt{2}).\frac{\sqrt{2}}{2} \quad \Rightarrow \quad x^2 = 7 - 4\sqrt{2} - 2\sqrt{2} + 2$$

$$\Rightarrow \quad x^2 = 5 - 2\sqrt{2} \quad \Rightarrow \quad x = \sqrt{5 - 2\sqrt{2}}$$

26. A regular polygon of side length $\sqrt{2-\sqrt{3}}$ is inscribed in a unit circle. What is the number of sides of this regular polygon?

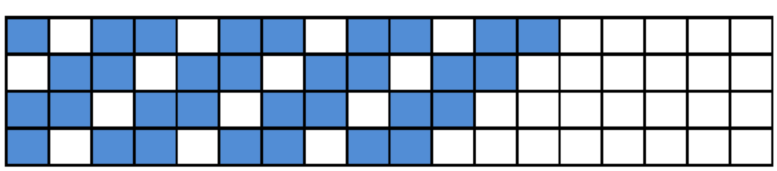
(a) 20 (b) 18 (c) 15 (d) 13 (e) 12

Consider the regular polygon inscribed in the unit circle with center O:
Let $OA = OB = OC = 1$. We also have $AB = BC = \sqrt{2-\sqrt{3}}$.



Let $m\angle AOB = m\angle BOC = x$
Writing Cosine Law for triangle ΔAOB : $AB^2 = AO^2 + BO^2 - 2AO \cdot BO \cos x$
$\Rightarrow 2 - \sqrt{3} = 1 + 1 - 2\cos x \Rightarrow \cos x = \sqrt{3}/2 \Rightarrow x = 30°$
Therefore, the number of sides of the regular polygon is 360/30 = 12.

27. A 4 by 503 chess board is colored in grey and white patterns as shown. How many grey squares are there of the total of 2012 squares of this chess board?

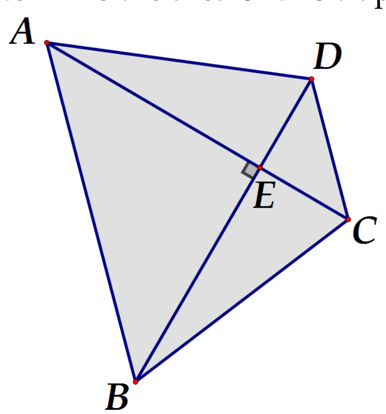


4 by 503

(a) 1343 (b) 1340 (c) 1341 (d) 1342 (e) 1344

For every 4 by 6 rectangle, there are 16 grey squares.
Dividing 503 by 6 yields a quotient of 83 and a remainder of 5.
Therefore, there will be $83 \cdot 16 + 13 = 1341$ grey squares.

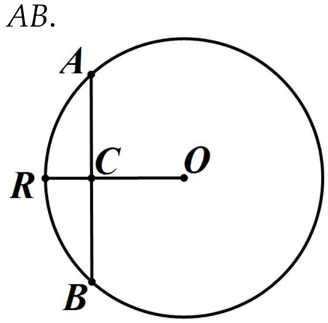
28. *ABCD* is an isosceles trapezoid with perpendicular diagonals and base lengths $DC = 2$ and $AB = 6$. Determine the area of this trapezoid.



(a) 14 (b) 16 (c) 18 (d) 20 (e) 22

Both ΔABE and ΔDEC are isosceles right triangles.
For ΔABE: The altitude drawn to side $\overline{AB}$ must be half length of side $\overline{AB}$.
For ΔDEC: The altitude drawn to side $\overline{DC}$ must be half length of side $\overline{DC}$.
Therefore, the altitude of the trapezoid is of length = 1 + 3 = 4
The area of the trapezoid = $\frac{2+6}{2} \cdot 4 = 16$

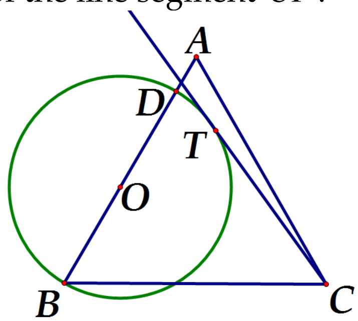
29. In a circle centered at *O*, radius $\overline{OR}$ is perpendicular to chord $\overline{AB}$ at point *C*. If $OC = 15$ and $CR = 2$, find *AB*.



(a) 8 (b) 10 (c) 12 (d) 14 (e) 16

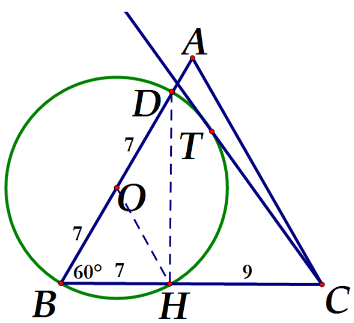
ΔACO is a right triangle with hypotenuse length $AO = OC + CR = 17$. Using the Pythagorean Theorem: $AC^2 + CO^2 = AO^2 \Rightarrow AC^2 + 15^2 = 17^2 \Rightarrow AC = 8$.
$AB = 2AC = 16$, because, in a circle, the perpendicular drawn from the center to any chord also bisects that chord.

30. *ABC* is an equilateral triangle with side length 16. The point *O* lies on line segment $\overline{AB}$. Ray $\overrightarrow{CT}$ is tangent at the point *T* to the circle of radius 7 centered at *O*. Determine the length of the line segment $\overline{CT}$.



(a) 10 (b) 11 (c) 12 (d) 13 (e) 14

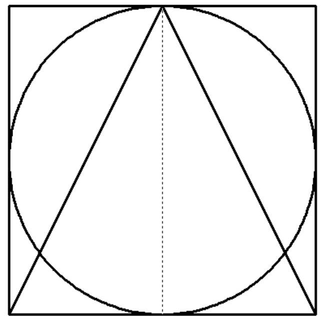
Create point *H*, intersection of the circle with segment $\overline{BC}$.



The triangle that forms, ΔDBH, is a 30-60-90 triangle, with $m\angle DHB = 90°$. Therefore *BH* = 7 as well. *BC* = 16, therefore, *HC* = 16 – 7 = 9. Using the power of point theorem yields:

$CT^2 = CH \cdot CB = 9 \cdot 16 = 144 \Rightarrow CT = 12$

31. The Greek mathematician Archimedes liked the design below so much that he wanted it on his tombstone.



When each of the figures is revolved about the vertical line of symmetry, it generates a solid of revolution – a cylinder, a sphere, or a cone. Find the ratio of volumes $V_{CYLINDER} : V_{SPHERE} : V_{CONE}$.

(a) $3:2:1$ (b) $\sqrt{3} : \frac{\pi}{\sqrt{2}} : 1$ (c) $4:\pi:2$ (d) $3 : \frac{2\pi}{3} : 1$ (e) $2\sqrt{\pi} : \frac{\pi}{\sqrt{2}} : \sqrt{\frac{3}{2}}$

Assume the square is a unit square. Using the volume formulas, we get:

$V_{CYLINDER} = \pi \cdot r^2 \cdot h = \pi \cdot 1^2 \cdot 2 = 2\pi$

$V_{SPHERE} = \frac{4}{3}\pi \cdot r^3 = \frac{4}{3}\pi \cdot 1^3 = \frac{4}{3}\pi$

$V_{CONE} = \frac{1}{3}\pi \cdot r^2 \cdot h = \frac{2}{3}\pi$

Therefore, the ratio of volumes is $3:2:1$.

32. Find the area of a square with the same perimeter as a regular hexagon of area $54\sqrt{3}$ square units.

(a) 9 (b) 36 (c) 81 (d) 144 (e) 225

Let a be the side length of the regular hexagon. Then solving the equation $6a^2 \frac{\sqrt{3}}{4} = 54\sqrt{3}$ for a yields $a = 6$. The perimeter of the hexagon = 36.
The square has the same perimeter; therefore, the side length of the square is 9 units.
The area of the square is then 81 square units.

33. In how many ways can Team USA's season of 10 soccer games result in 5 wins, 3 draws, and 2 losses in the 2014 World Cup qualification?

(a) 60 (b) 600 (c) 2520 (d) 59049 (e) 216000

$\frac{10!}{5!3!2!} = 2520$

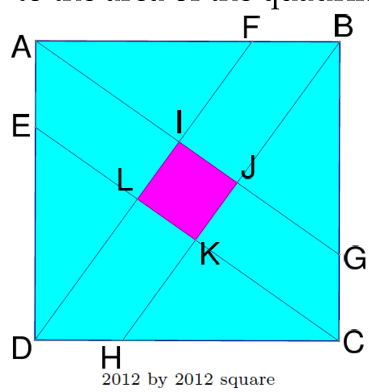
34. The Golden Ratio φ is a positive number that satisfies the quadratic equation $x^2 - x - 1 = 0$. By substitution, all positive integer powers of φ can be expressed in the form $a\varphi + b$. Determine the value of $a + b$ for φ^4.

(a) 3 (b) 5 (c) 8 (d) 13 (e) 21

$\varphi^4 = \varphi^2\varphi^2 = (\varphi+1)(\varphi+1) = \varphi^2 + 2\varphi + 1 = \varphi + 1 + 2\varphi + 1 = 3\varphi + 2$

Therefore $a + b = 5$

35. On square *ABCD* with side length 2012, points *E*, *F*, *G*, and *H* on each side are chosen in such a way that $AF = BG = CH = DE = 1509$. Determine the ratio of the area of the square *ABCD* to the area of the quadrilateral *IJKL*.

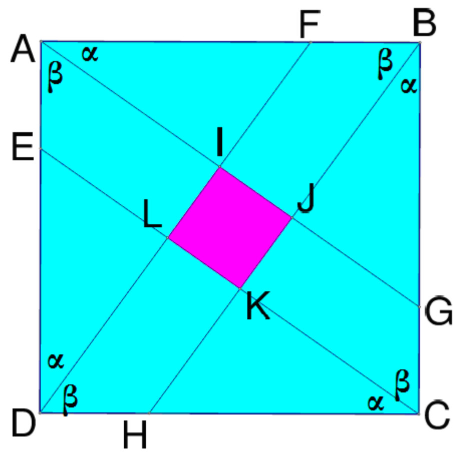


(a) 25 (b) 24 (c) 23 (d) 22 (e) 21

$\frac{AF}{FB} = \frac{BG}{GC} = \frac{CH}{HD} = \frac{DE}{EA} = \frac{1509}{503} = 3$

By symmetry, the four triangles ABJ, BCK, CDL, DAI are congruent.

Moreover, all these four triangles are right triangles. For instance, if $m\angle BAJ = \alpha$, then $m\angle DAI = \beta = 90 - \alpha$.



Hence all the angles at I, J, K, L are right angles.

If CL = a, CK = b, then LK = CL – CK = a – b. Similarly each of the other sides of IJKL equals a – b, hence it is a square.

AF = 3FB implies AI = 3IJ

Compute the area of square ABCD in two different ways:

$$AB^2 = 4\frac{AJ \cdot JB}{2} + JK^2$$

AJ = 4IJ = 4JK

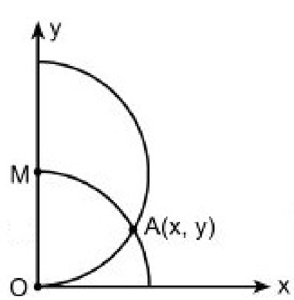
JB = 3JK

Substituting these in the above equation we get:

$$AB^2 = 4\frac{4JK \cdot 3JK}{2} + JK^2 = 25JK^2$$

Namely $\frac{AB^2}{JK^2} = 25$

36. A semicircle of radius 2 with center $M(0,2)$ and a quarter circle of radius 2 centered at the origin intersect at a point $A(x, y)$ in the first quadrant. Find the value of x.



(a) $\frac{5}{3}$ (b) $\sqrt{2}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{3}{2}$ (e) $\sqrt{3}$

By symmetry, the midpoint of $\overline{OM}$ has the same y-coordinate as point A. Because point M has coordinates $(0,2)$, the value of y must be 2/2 = 1. To determine the value of x, we can use the equation of either circle.
$x^2 + y^2 = 4 \Rightarrow x^2 = -y^2 + 4 = -1 + 4 = 3 \Rightarrow x = \pm\sqrt{3}$
Because A is in the first quadrant, $x = \sqrt{3}$.

37. A locus is the set of points that satisfy a given condition. Determine an equation of the locus of complex numbers $z = x + iy$ satisfying the equation $|z + 2 - i| = 10$.

(a) $(x-1)^2 + (y-1)^2 = 100$
(b) $(x-3)^2 + (y-1)^2 = 64$
(c) $(x+2)^2 + (y-1)^2 = 100$
(d) $(x-4)^2 + (y-1)^2 = 64$
(e) $(x-4)^2 + (y-4)^2 = 100$

$|z + 2 - i| = 10 \Rightarrow |(x+2) + i(y-1)| = 10 \Rightarrow (x+2)^2 + (y-1)^2 = 100$

38. Consider the exponential function $f(x) = a^x$ where a is a positive number different from 1. For what value of a does the slope of the tangent line to the graph of f at its y-intercept equal 1?

(a) 0.5 (b) $\sqrt{2}$ (c) 2 (d) e (e) π

This is the definition of natural exponential function, therefore, $a = e$.

39. Find the coordinates of the point on the circle $x^2 + y^2 = 4$ that has the minimum distance from the line $3x + 4y - 12 = 0$.

(a) $\left(\frac{6}{5}, \frac{8}{5}\right)$ (b) $\left(\frac{8}{5}, \frac{6}{5}\right)$ (c) $\left(\frac{12}{5}, \frac{16}{5}\right)$ (d) $\left(\frac{16}{5}, \frac{12}{5}\right)$ (e) $\left(\frac{5}{6}, \frac{4}{3}\right)$

Let $P\left(x, \pm\sqrt{4 - x^2}\right)$ be a point on the circle. Using point-line distance formula, the distance from P to the line $3x + 4y - 12 = 0$ is given by:

$$d=\frac{\left|3x\pm 4\sqrt{4-x^2}-12\right|}{\sqrt{3^2+4^2}}=\frac{\left|3x\pm 4\sqrt{4-x^2}-12\right|}{5}$$

Applying the sign test on the expression in the numerator yields

$$\left|3x\pm 4\sqrt{4-x^2}-12\right|=-3x\mp 4\sqrt{4-x^2}+12$$

In other words, sign test yields that the expression in absolute value is always negative. It will suffice to minimize this expression. Let

$$f(x)=-3x\mp 4\sqrt{4-x^2}+12$$

$$f'(x)=-3\pm\frac{4x}{\sqrt{4-x^2}}$$

Solving $f'(x)=0\Rightarrow 3=\pm\frac{4x}{\sqrt{4-x^2}}\Rightarrow 9=\frac{16x^2}{4-x^2}\Rightarrow 36-9x^2=16x^2\Rightarrow 36=25x^2\Rightarrow x=\pm\frac{6}{5}$.

Plugging these values in d above, we get:

$$d\left(\frac{6}{5}\right)=\frac{\left|3\cdot\frac{6}{5}\pm 4\sqrt{4-\left(\frac{6}{5}\right)^2}-12\right|}{5}=0.4 \text{ or } 2.96$$

$$d\left(-\frac{6}{5}\right)=\frac{\left|3\cdot\frac{6}{5}\pm 4\sqrt{4-\left(\frac{6}{5}\right)^2}-12\right|}{5}=1.84 \text{ or } 4.4$$

Namely d is a minimum for $x=\frac{6}{5}$. The corresponding y value is

$$+\sqrt{4-x^2}=+\sqrt{4-\left(\frac{6}{5}\right)^2}=\frac{8}{5}.$$

40. If $\cos x-\sin x=\frac{1}{2}$, what is the value of $\cos 2x$ if x is in the interval $\left[0,\frac{\pi}{2}\right]$?

(a) $\frac{\sqrt{7}}{4}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $-\frac{1}{4}$ (e) -1

$$\cos x-\sin x=\frac{1}{2}\Rightarrow(\cos x-\sin x)^2=\left(\frac{1}{2}\right)^2\Rightarrow 1-2\sin x\cos x=1/4\Rightarrow 2\sin x\cos x=3/4$$

$$\Rightarrow \sin 2x=3/4\Rightarrow \cos 2x=\sqrt{1-\sin^2 2x}=\sqrt{7/16}=\sqrt{7}/4$$

41. Compute $\left|\sum_{k=1}^{n} i^k\right|$ where $i=\sqrt{-1}$ and n is the largest perfect cube less than 100.

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

$\left|\sum_{k=1}^{4} i^k\right| = |i-1-i+1| = |0| = 0$

Therefore, $\left|\sum_{k=1}^{64} i^k\right| = 0$ as well.

42. Given that the real-valued function *f* satisfies the functional equation $f(x)-4f\left(\frac{1}{x}\right)=x$ for all $x>0$, find the minimum value of $|f(x)|$.

(a) $\frac{4}{15}$ (b) $\frac{7}{15}$ (c) $\frac{1}{2}$ (d) $\frac{4}{5}$ (e) $\frac{3}{5}$

$$\begin{cases} f(x)-4f\left(\frac{1}{x}\right)=x \\ f\left(\frac{1}{x}\right)-4f(x)=\frac{1}{x} \end{cases} \Rightarrow \begin{cases} f(x)-4f\left(\frac{1}{x}\right)=x \\ 4f\left(\frac{1}{x}\right)-16f(x)=\frac{4}{x} \end{cases} \Rightarrow -15f(x)=x+\frac{4}{x}$$

Now x and $4/x$ are two numbers with geometric mean 2, and arithmetic mean $\frac{-15f(x)}{2}$.
Solving the inequality $G.M. \le A.M.$ for $f(x)$ yields 4/15 as the minimum value of $|f(x)|$.

43. Find the sum $\sum_{k=1}^{2012}\left(\frac{1}{k\sqrt{k+1}+(k+1)\sqrt{k}}\right)$.

(a) $\sqrt{\frac{2012}{2013}}$ (b) $\frac{1}{\sqrt{2013}}$ (c) $\frac{\sqrt{2013}-1}{\sqrt{2013}}$ (d) $\sqrt{2013}$ (e) $\frac{\sqrt{2013}-\sqrt{2012}}{\sqrt{2013}}$

The summand can be written as $\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k+1}}$.

$$\sum_{k=1}^{2012}\left(\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k+1}}\right)=1-\frac{1}{\sqrt{2013}}=\frac{\sqrt{2013}-1}{\sqrt{2013}}$$

44. If $a > 1$, what is the value of the following limit?

$$\lim_{x\to 0}\frac{\sqrt[a]{1+ax}-\sqrt[a]{1-ax}}{x}$$

(a) a (b) $a+1$ (c) 2 (d) e (e) $2a$

We have the indeterminate form 0/0 as x goes to 0. Applying L'Hospital's Rule:

$$\lim_{x\to 0}\frac{\sqrt[a]{1+ax}-\sqrt[a]{1-ax}}{x}=\lim_{x\to 0}\frac{(1+ax)^{1/a}-(1-ax)^{1/a}}{x}=\lim_{x\to 0}\frac{(1/a)a(1+ax)^{-1+1/a}+(1/a)a(1-ax)^{-1+1/a}}{1}=2$$

45. Determine the value of the following limit.

$$\lim_{x\to\infty}\left(\frac{2x+5}{2x+3}\right)^{4x-1}$$

(a) 2 (b) 4 (c) e^2 (d) e^3 (e) e^4

$$\lim_{x\to+\infty}\left(\frac{2x+5}{2x+3}\right)^{4x-1} = \lim_{x\to+\infty}\left(1+\frac{2}{2x+3}\right)^{4x-1} = e^{\frac{2.4}{2}} = e^4$$

46. What is the value of the largest possible real number δ such that $\left|\sqrt{4x+1}-3\right| < 0.5$ whenever $|x-2| < \delta$?

(a) 3/8 (b) 5/8 (c) 7/8 (d) 11/16 (e) 13/16

$$\left|\sqrt{4x+1}-3\right| < 0.5 \Rightarrow 5/2 < \sqrt{4x+1} < 7/2 \Rightarrow 25/4 < 4x+1 < 49/4$$

$$\Rightarrow 21/4 < 4x < 45/4 \Rightarrow 21/16 < x < 45/16 \Rightarrow -11/16 < x-2 < 13/16 \Rightarrow |x-2| < 11/16$$

47. Let a, b, c be positive integers. How many ordered triples (a,b,c) satisfy the equation $a^2+b^2+c^2=2012$?

(a) 10 (b) 8 (c) 4 (d) 1 (e) 0

Every perfect square is a multiple of four or a multiple of four plus one ($(2k)^2 = 4k^2$, $(2k+1)^2 = 4k^2 + 4k + 1$). So, $2012 = 4(503) = x^2 + y^2 + z^2$ is a multiple of four only if all of the numbers x, y and z are even, say $x = 2a$, $y = 2b$ and $z = 2c$. So, the equation reduces to $503 = a^2 + b^2 + c^2$. Again since $503 = 4(125) + 3$ the numbers a, b and c must be all odd. Out of two consecutive integers one of them must be even, so $k(k+1)$ is an even number if k is an integer. But then, from the identity

$$(2k+1)^2 = 4k^2 + 4k + 1 = 4k(k+1) + 1, \ k \text{ integer}$$

we see that every odd perfect square is a multiple of 8 plus one. Therefore $a^2 + b^2 + c^2$ is a multiple of 8 plus 3.

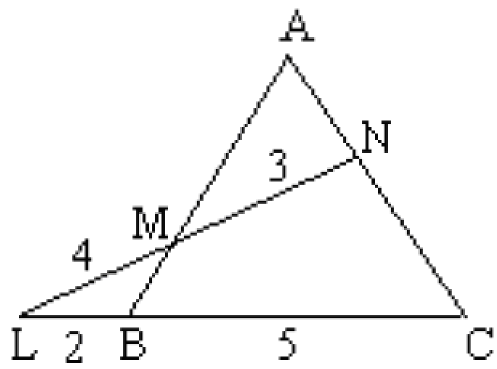
On the other hand, 503 ($8 \times 62 + 7$) is a multiple of 8 plus 7. This contradiction shows that $2012 = x^2 + y^2 + z^2$ is not possible for integers x, y and z.

48. Let $X = \left\{ x = \frac{n}{2} + \frac{m}{5} \middle| m, n \in \{0,1,2,\cdots,100\} \right\}$. Find the sum of the elements of X.

(a) 21395 (b) 22395 (c) 23395 (d) 24395 (e) 25395

$\frac{n}{2} + \frac{m}{5} = \frac{5n+2m}{10}$. Let $y = 5n + 2m$. $\sum x = \frac{\sum y}{10}$ such that $y \in \{0,1,2,\cdots,700\}$. But y cannot assume all these values. For instance, it is impossible to express 1 in the form $5n + 2m$. Similarly, it is impossible to express 3, 699, and 697 in this form. Therefore, the sum of the elements of X is $\frac{\sum_{k=0}^{700} k - (1+3+697+699)}{10} = \frac{1}{10}\left(\frac{700 \cdot 701}{2} - 1400\right) = 24395$

49. ABC is a triangle with $BC = 5$. A line crosses lines BC, AC and AB at points L, N and M respectively, such that $LM = 4, MN = 3, LB = 2$. Determine the ratio $\frac{NA}{NC}$.

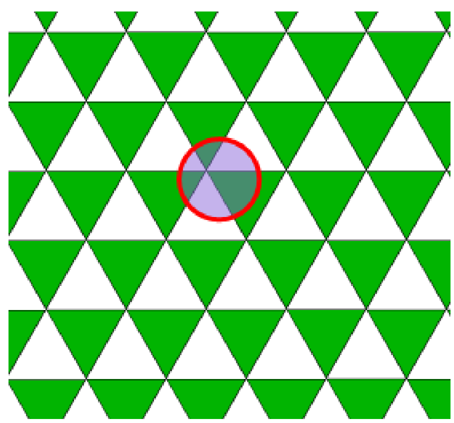


(a) 3/7 (b) 7/15 (c) 6/17 (d) 4/15 (e) 4/21

Using Menelaus' Theorem:

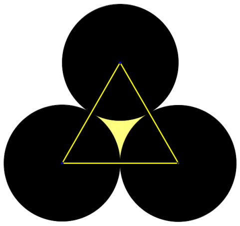
$$\frac{|AN|}{|AC|}\cdot\frac{5}{2}\cdot\frac{4}{3}=1 \Rightarrow \frac{|AN|}{|AC|}=\frac{6}{20}=\frac{3}{10} \Rightarrow \frac{|NA|}{|NC|}=\frac{3}{7}$$

50. The figure below shows a tessellation of the plane with equilateral triangles of side length 1. In the experiment of tossing a coin of diameter 1, what is the probability that the coin does not overlap with any triangle vertex?



(a) $\frac{8\sqrt{3}-4\pi}{7\sqrt{3}}$ (b) $\frac{8\sqrt{3}-4\pi}{8\sqrt{3}}$ (c) $\frac{8\sqrt{3}-4\pi}{9\sqrt{3}}$ (d) $\frac{8\sqrt{3}-4\pi}{10\sqrt{3}}$ (e) $\frac{8\sqrt{3}-4\pi}{11\sqrt{3}}$

Consider one of the equilateral triangles:



The only way the coin will not overlap with any triangle vertex will happen if the coin lands in the yellow region shown in the diagram above. Therefore, it suffices to determine the area of this yellow region:

The three sectors are each of 60 degree central angle, summing to a 180 degree central angle, i.e., a semicircle. Therefore, the area of the yellow region equals = the area of the equilateral triangle of side length 1 – the area of a semicircle of radius 1/2.

$$= 1^2 \cdot \frac{\sqrt{3}}{4} - \frac{\pi \cdot (1/2)^2}{2} = \frac{\sqrt{3}}{4} - \frac{\pi}{8}$$

Dividing this value by the area of the equilateral triangle will give us the desired probability:

$$P = \frac{\frac{\sqrt{3}}{4} - \frac{\pi}{8}}{\frac{\sqrt{3}}{4}} = 1 - \frac{\pi}{8} \cdot \frac{4}{\sqrt{3}} = \frac{8\sqrt{3} - 4\pi}{8\sqrt{3}}$$