Thirty-fifth Annual Columbus State Invitational Mathematics Tournament

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Department of Mathematics
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Solutions to the 2009 Mathematics Tournament Problems

1) Jack had an average score of 85 on his first 4 quizzes. He has an average score of 82 on his first 5 quizzes. What did he receive on his fifth test?

A) 60 B) 65 C) 66 D) 70 E) 75

D) Let q_i be the score of quiz i. Since $\frac{q_1+q_2+q_3+q_4}{4}=85$,

$q_1+q_2+q_3+q_4=340$. Similarly, $\frac{q_1+q_2+q_3+q_4+q_5}{5}=82$ implies that

$q_1+q_2+q_3+q_4+q_5=340+q_5=410$. Therefore, $q_5=70$.

2) For $x\neq 0,\pm 1$, the expression $\dfrac{\frac{1}{x^{2007}}-\frac{1}{x^{2009}}}{\frac{1}{x^{2008}}-\frac{1}{x^{2010}}}$ is equivalent to which of the following?

A) x B) $x-1$ C) x^2-1 D) $\frac{1}{x}$ E) 1

A) $$\dfrac{\frac{1}{x^{2007}}-\frac{1}{x^{2009}}}{\frac{1}{x^{2008}}-\frac{1}{x^{2010}}}=\frac{x^3-x}{x^2-1}=\frac{x(x^2-1)}{x^2-1}=x.$$

3) Find the product of the solutions of the equation $\frac{3}{x}=\frac{8}{x-3}-1$.

A) -6 B) -9 C) 9 D) $-\frac{9}{5}$ E) 10

B) Multiplying each side of the equation by $x(x-3)$, we obtained the equation $x^2-8x-9=0$ with the solutions $x=9$ and $x=-1$.

4) If the surface area of cube A is 64% of the surface area of cube B, then the volume of cube A is what percentage of the volume of cube B?

A) 0.64 B) 0.512 C) 51.2 D) 32 E) 64

C) Let a and b be the side length of cube A and cube B respectively. Then $\frac{6a^2}{6b^2}=0.64$. So $\frac{a}{b}=0.8$, which gives $\frac{a^3}{b^3}=0.512=51.2\%$.

5) If the two lines $3y + x + 2 = 0$ and $2y + ax + 3 = 0$ are perpendicular, what is the value of a?

A) $\frac{1}{3}$ B) $-\frac{1}{3}$ C) 6 D) -6 E) $\frac{3}{2}$

D) The first line has the slope $-\frac{1}{3}$ and the second line has the slope $-\frac{a}{2}$. Since two lines are perpendicular, $-\frac{a}{2} = -\frac{1}{-\frac{1}{3}}$. Thus $a = -6$.

6) What is the solution set for the inequality $2x^2 + x < 6$?

A) $-2 < x < \frac{3}{2}$ B) $x > \frac{3}{2}$ or $x < -2$ C) $x < \frac{3}{2}$

D) $\frac{3}{2} < x < 2$ E) $x < -2$

A) $2x^2 + x - 6 < 0$ or $(2x-3)(x+2) < 0$. The parabola opens up. The solution is between the two zeros of the parabola, -2 and $\frac{3}{2}$.

7) The parabola $y = x^2 - bx + 4$ has its vertex on the x-axis. What are the values of b?

A) ± 4 B) ± 5 C) ± 6 D) ± 7 E) ± 8

$$y = x^2 - bx + 4 = x^2 - bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + 4 = \left(x - \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + 4.$$

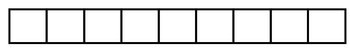
A) The vertex is $\left(\frac{b}{2}, -\left(\frac{b}{2}\right)^2 + 4\right)$. Since the parabola has the vertex on the x-axis, $-\left(\frac{b}{2}\right)^2 + 4 = 0$. So $b = \pm 4$.

8) Define the function $f(x) = \frac{1}{x-1}$. Find $f(f(f(x)))$.

A) $\frac{x-1}{2-x}$ B) $\frac{x-2}{x-3}$ C) $\frac{2-x}{2x-3}$ D) $\frac{x-3}{2-x}$ E) $\frac{2-x^2}{x^2-3}$

C) $f(f(x)) = \dfrac{1}{\dfrac{1}{x-1}-1} = \dfrac{x-1}{2-x}$, $f(f(f(x))) = \dfrac{1}{\dfrac{x-1}{2-x}-1} = \dfrac{2-x}{2x-3}$.

9) Each of the 9 squares shown contains one number chosen from 1, 2, 3, 4, 5, 6, 7, 8, and 9. No number is repeated. Suppose that the sum of the numbers in the first five squares is 35 and that the sum of the numbers in last five squares is 16. What number goes to the fifth square?



A) 4 B) 5 C) 6 D) 7 E) 8

C) Let x be the number in the fifth square. Adding the sum of the numbers in the first five squares and the sum of the numbers in last five squares gives the equation

$$35+16=1+2+3+4+5+6+7+8+9-x$$

since x is counted twice. Thus $x=6$.

10) How many diagonals does a regular dodecagon have? A dodecagon is a polygon with twelve vertices.

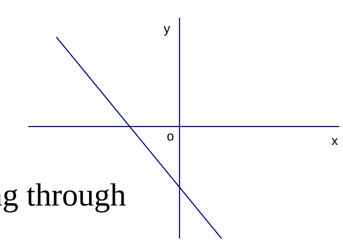
A) 108 B) 132 C) 144 D) 84 E) 54

E) Diagonals are formed by connecting one vertex to other vertices that are not adjacent to that vertex. Note that each vertex is used twice if all 12 vertices are used as starting vertex in the construction. Therefore, the number of diagonals is $\frac{1}{2}(12)(9)=54.$

11) If $ab>0$ and $ac>0$, then the line $ax+by+c=0$ can not pass through which of the following quadrants?

A) Quadrant I B) Quadrant II C) Quadrant III
D) Quadrant IV E) Quadrant I and Quadrant II

A) The x-intercept of the line is $x=-\frac{c}{a}$, which is negative since $ac>0$. The slope of the line is $m=-\frac{a}{b}$, which is also negative since $ab>0$. Therefore, the line is decreasing and passing through Quadrants II, III, and IV only.



12) If the roots of $x^3+kx^2+hx-2009=0$ are $a, b,$ and c, find $a^2+b^2+c^2$.

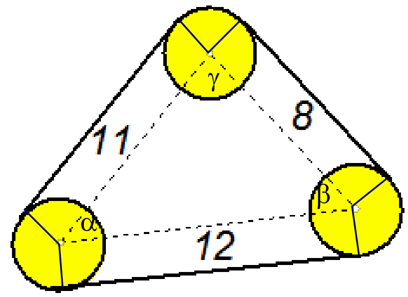
A) $k^2 - 2h$ B) $k - 2009$ C) $k^2 - 4018$ D) $\frac{k}{h}$ E) $2009 - h$

A) Note that $a+b+c=-k$ and $ab+ac+bc=h$. So

$$k^2=(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc,$$

$$a^2+b^2+c^2=k^2-2h.$$

13) Circles of radius 2 are centered at the vertices of a triangle with sides lengths 8, 11, and 12. Find the length of a belt that fits tightly around those three circles as shown in the figure.



A) 18.43 B) $31+4\pi$ C) $31+6\pi$

D) $31+\sqrt{3}\pi$ E) $31+\sqrt{2}\pi$

B) Draw perpendicular lines from the centers of the circles to the belt. It can be seen that the length of the belt is sum of 8, 11, 12 and the tree arc lengths around the circles. The central angles are $\pi-\alpha, \pi-\beta$, and $\pi-\gamma$, respectively. So the length of the belt is

$$8+11+12+2(\pi-\alpha)+2(\pi-\beta)+2(\pi-\gamma)=31+6\pi-2(\alpha+\beta+\lambda)$$

$$=31+6\pi-2\pi=31+4\pi$$

14) $n=2009^2-2008^2+2007^2-2006^2+\cdots+5^2-4^2+3^2-2^2$ is not divisible by which of the following?

A) 4 B) 251 C) 1004 D) 2011 E) 512

E)

$2009^2-2008^2+2007^2-2006^2+\cdots+5^2-4^2+3^2-2^2$

$=(2009-2008)(2009-2008)+(2007-2006)(2007+2006)+\cdots+(3-2)(3+2)$

$=4017+4013+4009+\cdots+5=\dfrac{(5+4017)(1004)}{2}=2019044$

Factor or direct dividing. 512 won't work.

15) Suppose that $A \neq 0$ and $Ax^3+Ax^2+Bx+C=0$ for $x=\pm 1$. Which of the following is a root of multiplicity 2 of the equation?

A) 1 B) -1 C) 0 D) 2 E) -2

B) Substituting x for 1 and -1 yields two equations

$$A + A + B + C = 0,$$
$$-A + A - B + C = 0.$$

So we have $B = -A$ and $C = -A$. The equation $Ax^3 + Ax^2 + Bx + C = 0$ becomes $Ax^3 + Ax^2 - Ax - A = 0$ or $(x+1)^2(x-1) = 0$ since $A \neq 0$.

16) The number M has 1111 digits, each of which is equal to 1, $M = 111\ldots111$. What is the sum of the digits of $1001 \times M$?

A) 1111 B) 1221 C) 1122 D) 2112 E) 2222

E) $1001 \times M = 1000 \times M + M$. The digits are 6 ones and 1108 twos. The sum of the digits of $1001 \times M = 2222$.

17) What are the last 3 digits of the number 3^{98}?

A) 998 B) 899 C) 999 D) 989 E) 889

E)

$$3^{98} = 9^{49} = (10-1)^{49}$$
$$= 10^{49} - \binom{49}{1}10^{48} + \cdots + \binom{49}{46}10^3(-1)^{46} + \binom{49}{47}10^2(-1)^{47} + \binom{49}{48}10(-1)^{48} + (-1)^{49}$$
$$= 1000M - 117600 + 490 - 1 = 1000M - 117111$$

So the last three digits are 889.

18) What is the coefficient of a^{-1003} in the binomial expansion of $\left(a - \frac{1}{\sqrt{a}}\right)^{2009}$?

A) 2009 B) -2009 C) 2017036 D) -2017036 E) 2008

A) The nth term in the expansion is

$$\binom{2009}{n} a^n \left(-\frac{1}{\sqrt{a}}\right)^{2009-n} = (-1)^{2009-n}\binom{2009}{n} a^{n-\frac{2009-n}{2}}$$

If $n - \frac{2009-n}{2} = -1003$, then $n = 1$. So

$$(-1)^{2009-n}\binom{2009}{n} = (-1)^{2009-1}\binom{2009}{1} = 2009.$$

19) For $a, b > 1$, when simplified, what does the expression $(\log b)(\log_b a) \div \log\left(\frac{1}{a}\right)$ become?

A) 1 B) -1 C) $-\frac{1}{2}$ D) $\frac{1}{2}$ E) 2

B) $\log b \log_b a \div \log \frac{1}{a} = \frac{\log b\left(\frac{\log a}{\log b}\right)}{-\log a} = -1.$

20) What is the range of the function $y = \left(\frac{\cos^{-1}(3x-1)}{\pi} + 1\right)^2$ with the domain $[0, 2/3]$?

A $[0, \infty)$ B) $[1, 4]$ C) $[0, \pi]$ D) $[1, \pi]$ E) $[0, \pi^2]$

B) The range of $y = \cos^{-1}(3x-1)$ is $[0, \pi]$. So the rang of the function is $[1, 4]$.

21) If $8\tan x = 3\cos x$, what is the value of $\sin x$?

A) -3 B) $\frac{1}{3}$ C) $-\frac{1}{3}$ D) -1 E) $\frac{3}{8}$

B) $8\left(\frac{\sin x}{\cos x}\right) = 3\cos x$. So $3\sin^2 x + 8\sin x - 3 = 0$, which gives $\sin x = \frac{1}{3}, -3$. But since $-1 \le \sin x \le 1$, $\sin x = \frac{1}{3}$.

22) Write the polar equation $r = \csc\left(\theta + \frac{\pi}{3}\right)$ in rectangular form with y expressed as a function of x.

A) $y = \sqrt{3}x - 2$ B) $y = \sqrt{3}x + 2$ C) $y = -\sqrt{3}x - 2$

D) $y = -\sqrt{3}x + 2$ E) None of these

D) Since $\cos\theta = \frac{1}{\sin\theta}$, we rewrite $r = \csc\left(\theta + \frac{\pi}{3}\right)$ as $r\sin\left(\theta + \frac{\pi}{3}\right) = 1$. Using the sum formula, we have $r\sin\theta\cos\frac{\pi}{3} + r\cos\theta\sin\frac{\pi}{3} = 1$ or $y = -\sqrt{3}x + 2$.

23) Let $A=\sqrt{x+\sqrt{6x-9}}$ and $B=\sqrt{x-\sqrt{6x-9}}$. Find the real value(s) of x so that $A+B=\sqrt{6}$ and A and B are real numbers.

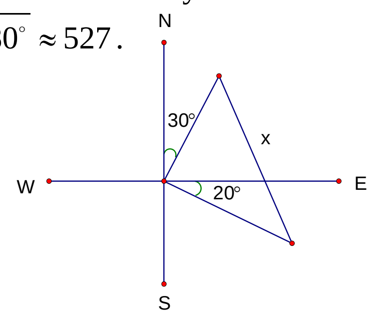
A) 3 B) 0, 3 C) $\left[\frac{3}{2},3\right]$ D) $(-\infty,3]$ E) $[3,\infty)$

C) Squaring both sides of the equation $A+B=\sqrt{6}$ yields an equation $x+|x-3|=3$. If $x\geq 3$, the equation becomes $x+x-3=3$ and we have a solution $x=3$. If $x<3$, the equation becomes $x-x+3=3$, which indicates that the original equation is an identity for $x<3$. Since each square root is a real number, we must have $6x-9\geq 0$ or $x\geq\frac{3}{2}$. Note that $x-\sqrt{6x-9}\geq 0$ for all x. Therefore, the solution set for the equation is $\left[\frac{3}{2},3\right]$.

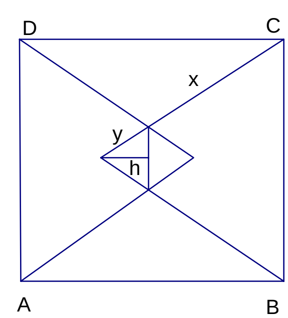
24) Two planes left the airport at the same time. One flew 200 mph at a bearing $N30^\circ E$ (30° east of north) and other at 210 mph at a bearing $E20^\circ S$ (20° south of east). To the nearest mile, how far apart are they after 2 hours?

A) 500 miles B) 527 miles C) 528 miles D) 529 miles E) 530 miles

B) Let x be the distance between the two planes after two hours. By the law of cosine, we have $x=\sqrt{400^2+420^2-2\cdot 400\cdot 420\cos 80^\circ}\approx 527$.



25) Two equilateral triangles ΔADE and ΔBGC are inside a square $ABCD$ as shown in the figure. Find the ratio of the shaded area to the area of the square.



A) $\frac{7\sqrt{3}}{6}-2$ B) $\frac{7\sqrt{3}}{8}-\frac{3}{2}$ C) $\frac{5\sqrt{3}}{3}-1$
D) $\frac{2\sqrt{3}}{3}-\frac{1}{2}$ E) $\frac{2\sqrt{3}}{3}-1$

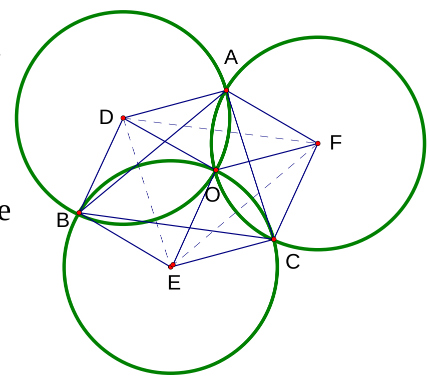
E) Let $2a$ be the side length of the square. Let $x = CF$, $y = GF$, and h be the height of ΔGHF. $x = \frac{a}{\cos 30^\circ} = \frac{2a}{\sqrt{3}}$, $y = 2a - \frac{2a}{\sqrt{3}}$, and $h = y\sin 60^\circ = \sqrt{3}a - a$. The shaded area is $2\left(\frac{1}{2}yh\right) = 2a^2\frac{4-2\sqrt{3}}{\sqrt{3}}$. Thus the ratio of the shaded area to the area of the square is

$$\frac{2a^2\frac{4-2\sqrt{3}}{\sqrt{3}}}{(2a)^2} = \frac{2-\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}-3}{3} = \frac{2\sqrt{3}}{3} - 1.$$

26) Three circles of the same radius 2 intersect at a point O and each two intersect at A, B, and C respectively. What is the radius of the circle that circumscribes ΔABC?

A) 1 B) 1.5 C) 2 D) 2.5 E) 3

C) Using rhombuses FOEC, ADOF, and BEOD, we see that $CF \| BD, AF \| BE,$ and $BD \| CF$ and all these sides are equal to 2. Note that ΔABC and ΔDEF are congruent and that the circle circumscribed to ΔDEF has a radius of 2. Therefore the circle circumscribed to ΔABC has a radius of 2.

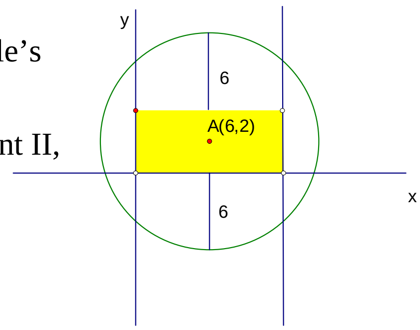


27) If a and b are randomly chosen integers from -3 to 3 inclusive, what is the probability that $x^2 + ax + b = 0$ has real solutions?

A) $\frac{13}{49}$ B) $\frac{17}{49}$ C) $\frac{28}{49}$ D) $\frac{31}{49}$ E) $\frac{34}{49}$

E) Use discriminant $a^2 - 4b \geq 0$ or $a^2 \geq 4b$, which holds when (a,b) are $(0,0),(\pm1,0),(\pm2,0),(\pm3,0)$, $(\pm2,1),(\pm3,1)$, $(\pm3,2)$, and $-3 \leq a \leq 3, -3 \leq b < 0$. So the probability is $\frac{34}{49}$.

28) The circle $(x-6)^2+(y-2)^2=64$ lies in all four quadrants. I_1 is the area of the portion of the circle's interior that is in Quadrant I, I_2 is the area of the portion of the circle's interior that lies in Quadrant II, I_3, in Quadrant III, and I_4, in Quadrant IV. Find $I_1 - I_2 + I_3 - I_4$.



A) $48(\sqrt{7}-1)$ B) 24 C) $30\sqrt{3}-10$
D) 48 E) $64\pi - 30$

D) From the shown picture and by symmetry, we see that $I_1 - I_2 + I_3 - I_4$ is the area of the shaded rectangle with length of 12 and width of 4. So
$I_1 - I_2 + I_3 - I_4 = 12 \cdot 4 = 48$.

29) If x and y are positive integers and $2xy = 2009 - 3y$, then how many positive solutions (x, y) are there?

A) 1 B) 2 C) 3 D) 4 E) 5

E) $2xy = 2009 - 3y$ yields $2xy + 3y = 2009$ or $y = \dfrac{2009}{2x+3}$. So $2x+3$ must be the factors of 2009. Since $2009 = 7^2 \cdot 41$. Thus $2x+3$ can be one of the factors 7, 49, 41, 287, and 2009.

30) An arbitrary triangle of perimeter 10 is formed. A second triangle is formed by joining the midpoints of the first triangle. A third triangle is formed by joining the midpoints of the second triangle, and so on indefinitely. Find the total length of all line segments in the resulting configuration.

A) 15 B) 20 C) 30 D) 100 E) ∞

B) The lengths of the triangles are a geometric sequence $10, \frac{10}{2}, \frac{10}{2^2}, \frac{10}{2^3}, \ldots$. The sum of those numbers is $\dfrac{10}{1-\frac{1}{2}} = 20.$

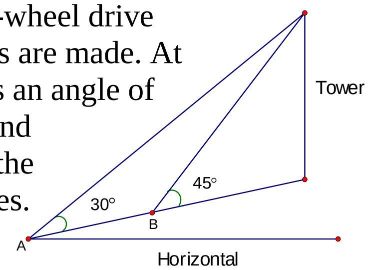
31) What is the value of the following product?

$$\sin\left(\frac{\pi}{2^{2009}}\right)\cos\left(\frac{\pi}{2^{2009}}\right)\cos\left(\frac{\pi}{2^{2008}}\right)\cos\left(\frac{\pi}{2^{2007}}\right)\cos\left(\frac{\pi}{2^{2006}}\right)\cdots\cos\left(\frac{\pi}{2^{3}}\right)\cos\left(\frac{\pi}{2^{2}}\right)$$

A) $\frac{1}{2^{2010}}$ B) $\frac{1}{2^{2009}}$ C) $\frac{1}{2^{2008}}$ D) 1 E) 0

C) Let A be the expression. Multiplying A by 2^{2008} and using $2\sin\theta\cos\theta = \sin 2\theta$, we have $2^{2008}A = \sin\frac{\pi}{2} = 1$. $A = \frac{1}{2^{2008}}$.

32) A 750-foot tall vertical tower is located at the top of a road. The road has a constant inclination. (See the diagram right.) A 4-wheel drive vehicle is driven up the road and two observations are made. At the first observation point (A), the tower subtends an angle of 30 degrees from the surface of the road. The second observation at point (B) is 500 feet further along the road and the tower subtends an angle of 45 degrees. Determine the inclination of the road.

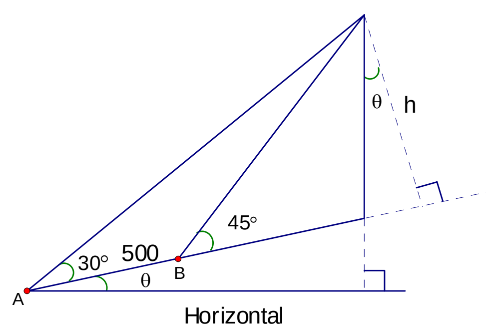


A) 20° B) 24° C) 24.4° D) 25° E) 15°

C) Drop a perpendicular from the top of the tower to the road. Let the perpendicular distance from the top of the tower to the road be h.

$$h\cot(30^\circ) - h\cot(45^\circ) = 500$$

$$h = \frac{500}{\sqrt{3}-1}$$



$$\theta = \arccos\left(\frac{h}{750}\right) = \arccos\left(\frac{500}{750(\sqrt{3}-1)}\right) \approx 24.4^\circ$$

33) Suppose that $f(x) = ax^2 + bx + a$ satisfies the equation $f\left(x+\frac{7}{4}\right) = f\left(\frac{7}{4}-x\right)$ and the equation $f(x) = 7x + a$ has only one solution. What is the value of $a+b$?

A) 3 B) 4 C) 5 D) 6 E) 7

C) Using $f\left(x+\frac{7}{4}\right)=f\left(\frac{7}{4}-x\right)$, we have $b=-\frac{7}{2}a$. So $f(x)=ax^2-\frac{7}{2}ax+a$.

Since the equation $f(x)=7x+a$ has only one solution, $\Delta=\left(\frac{7}{2}a+7\right)^2-4a\cdot 0=0$.

We have $a=-2$. Therefore, $b=7$ and $a+b=5$.

34) How many pairs of positive integers (a,b) with $a+2b<80$ satisfy the equation

$$a+\frac{1}{b}=7\left(b+\frac{1}{a}\right)?$$

A) 7 B) 8 C) 9 D) 10 E) 11

B) $a+\frac{1}{b}=7\left(b+\frac{1}{a}\right)$ implies that $\frac{ab+1}{b}=\frac{7(ab+1)}{a}$ or

$ba^2+(1-7b^2)a-7b=(ba+1)(a-7b)=0$. So $a=7b$. So $9b<80$. Thus $b<9$. There are 8 pairs.

35) For every pair of positive integers a and b, we consider the operation "$\#$" on positive integers with the following three properties:

a) $a\#a=a+2$
b) $a\#b=b\#a$
c) $\frac{a\#(a+b)}{a\#b}=\frac{a+b}{b}$

What is $3\#7$?

A) 7 B) 21 C) 49 D) 63 E) 77

D)

$$3\#7=\frac{3\#(3+4)}{3\#4}3\#4=\frac{3+4}{4}3\#4=\frac{7}{4}\frac{3\#(3+1)}{3\#1}3\#1=\frac{7}{4}\frac{3+1}{1}3\#1$$

$$=7(1\#3)=7\frac{1\#(1+2)}{1\#2}1\#2=7\frac{1+2}{2}\frac{1\#(1+1)}{1\#1}1\#1=\frac{21}{2}\frac{1+1}{1}(1+2)=63$$

36) How many distinct pairs of integers (x,y) satisfy the equation $x^2+y^2=2009$?

A) 2 B) 4 C) 6 D) 8 E) 10

D) The equation is $x^2 + y^2 = 7^2 41$. We use the classic result in Number Theory: Every prime p of the form $4k+1$ that divides $x^2 + y^2$ divides both x and y. This means that $x = 7a$ and $y = 7b$. The equation $x^2 + y^2 = 2009$ becomes $a^2 + b^2 = 41$. It is easy to check that the solutions of this equation are $(\pm4, \pm5)$ and $(\pm5, \pm4)$. Thus we get 8 solutions for (x, y).

37) A store has objects that cost either 10, 25, 50, or 70 cents. If Sharon buys 40 objects and spends seven dollars, what is the largest quantity of the 50 cent items that could have been purchased?

A) 3 B) 4 C) 5 D) 6 E) 7

D) Let a, b, c and d represent the quantities of items purchased for 10 cents, 25 cents, 50 cents, and 70 cents respectively. Then a, b, c and d are non-negative integers satisfying the following equations.

$$a + b + c + d = 40$$

$$10a + 25b + 50c + 70d = 700$$

Eliminating the variable a producing the following equation in a, b and c.

$$15b + 40c + 60d = 300$$

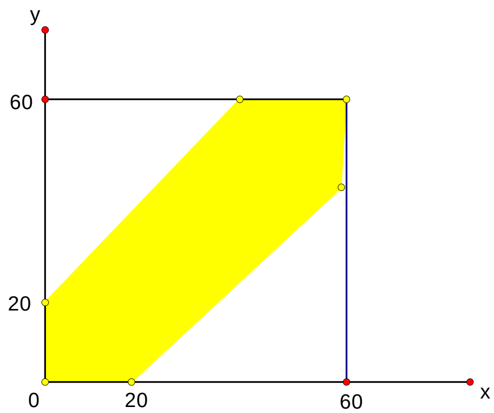
$$b + \frac{8}{3}c + 4d = 20$$

Since the right hand side of the equation is 20, c must be divisible by 3. The only possible values for c are 0, 3, and 6. c can not be 9 or more since $\frac{8}{3} \times 9 = 24 > 20$. But c can be 6 in several different ways. One of them is $c = 6, d = 0, b = 4,$ and $a = 30$.

38) Two friends agree to meet at library between 1:00 P.M. and 2:00 P.M. Each agrees to wait 20 minutes for the other. What is the probability that they will meet if their arrivals occur at random during the hour and if the arrival times are independent?

A) $\frac{3}{7}$ B) $\frac{4}{5}$ C) $\frac{4}{9}$ D) $\frac{5}{9}$ E) $\frac{7}{9}$

D) On the following diagram calculate $P(|x - y| < 20)$, which is the ratio of the shaded area to the area of the square. The shaded area is $60^2 - 2\left(\frac{1}{2}40^2\right)$.



$$P(|x-y|<20)=\frac{5}{9}$$

39) If $\tan x = 3\tan y$ for $0 \le y < x < \frac{\pi}{2}$, what is the maximum value of $x-y$?

A) $\frac{\pi}{6}$ B) $\frac{\pi}{4}$ C) $\frac{\pi}{3}$ D) $\frac{\pi}{12}$ E) $\frac{5\pi}{12}$

A) Let $u = x-y$. Then $0 < u < \frac{\pi}{2}$.

$\tan u = \tan(x-y) = \frac{\tan x - \tan y}{1+\tan x \tan y} = \frac{2\tan y}{1+3\tan^2 y} \le \frac{2\tan y}{2\sqrt{3\tan^2 y}} = \frac{1}{\sqrt{3}}$, where we used the inequality $a^2+b^2 \ge 2ab$ since $(a-b)^2 \ge 0$. Note that $\tan u$ obtains its maximum value $\frac{1}{\sqrt{3}}$ when $u = \frac{\pi}{6}$ and that when $\tan u$ reaches its maximum and u also reaches its maximum since $f(u) = \tan u$ is an increasing function for $0 < u < \frac{\pi}{2}$.

When $x = \frac{\pi}{3}$ and $y = \frac{\pi}{6}$, $u = x - y = \frac{\pi}{6}$.

40) Sally walked 4 miles north, then 2 miles east, 1 mile south, $\frac{1}{2}$ mile west, $\frac{1}{4}$ mile north, and so on. If she continued this pattern indefinitely, how far from her initial point will she approach?

A) 3 miles B) 3.2 miles C) 4 miles D) $\frac{13\sqrt{5}}{8}$ miles E) $\frac{8\sqrt{5}}{5}$ miles

E) Assume that Sally started at the origin of the $xy-$plane. The points at each turn were

$$(0,4),(2,4),(2,3),\left(2-\frac{1}{2},3\right),\left(2-\frac{1}{2},3+\frac{1}{2^2}\right),\left(2-\frac{1}{2}+\frac{1}{2^3},3+\frac{1}{2^2}\right),$$

$$\left(2-\frac{1}{2}+\frac{1}{2^3},3+\frac{1}{2^2}-\frac{1}{2^4}\right),\left(2-\frac{1}{2}+\frac{1}{2^3}-\frac{1}{2^5},3+\frac{1}{2^2}-\frac{1}{2^4}\right)\ldots$$

The destination point is

$$(2-(\frac{1}{2}+\frac{1}{2^5}+\frac{1}{2^9}\cdots)+(\frac{1}{2^3}+\frac{1}{2^7}+\frac{1}{2^{11}}\cdots),3+(\frac{1}{2^2}+\frac{1}{2^6}+\frac{1}{2^{10}}+\cdots)-(\frac{1}{2^4}+\frac{1}{2^8}+\frac{1}{2^{12}}+\cdots))$$

$$=(\frac{24}{15},\frac{16}{5})$$

$$\sqrt{\left(\frac{24}{15}\right)^2+\left(\frac{16}{5}\right)^2}=\frac{8\sqrt{5}}{5}$$

41) The number $2008ab2009$ is divisible by 99, where a and b denote two missing digits in base 10. What is the value of $(a^2+1)(b^2+1)+2a+3b$?

A) 98 B) 99 C) 100 D) 101 E) 102

B) Since $2008ab2009\equiv 0 \pmod{99}$ and

$$2008ab2009=2008ab0000+2009=2008ab0000+20\times 99+29,$$

we have $2008ab0000\equiv -29 \pmod{99}$. Furthermore,

$$2008ab0000=2008ab\times(9999+1)=2008ab\times 9999+2008ab$$

and $2008ab\times 9999$ is divisible by 99. It implies that $2008ab\equiv -29 \pmod{99}$.
Note that $2008ab=200800+ab=2028\times 99+28+ab$. So $ab\equiv -57 \pmod{99}$, i.e., $ab+57$ is integer multiple of 99. Using the fact that ab is a positive two digit integer, we conclude that $ab=42$ and $a=4, b=2.$ Therefore, $(a^2+1)(b^2+1)+2a+3b=99.$

Another way to do this problem is to use the tests for divisibility by 9 and by 11. Since $2008ab2009$ is divisible by 9, we have $21+a+b=9n$ for some nonnegative integer $n.$ Since $2008ab2009$ is also divisible by 11, we have $a-b-13=11m$ for some integer $m.$ Or $a+b=-3+9n$ and $a-b=2+11m.$ The fact that a and b are integers between 0 and 9 inclusive forces $n=1$ or $n=2$ and $m=0$ or $m=-1$.
If $m=-1$, the equation $a-b=2+11m$ indicates that $a=0$ and $b=9$ but $2008ab2009$ is NOT divisible by 9. Similarly, n can not be 2. Using $n=1$ and $m=0$, we obtain $a=4$ and $b=2.$

42) The altitudes of a triangle are 210, 195 and 182. What is the area of the triangle?

A) 22183 B) 22183.25 C) 22182 D) 22181 E) 22181.25

E) Let $a, b,$ and c be the three sides of the triangle with altitudes 210, 195 and 182, respectively. Then the area of the triangle is $A = \frac{1}{2}a \cdot 210 = \frac{1}{2}b \cdot 195 = \frac{1}{2}c \cdot 182$.

So $b == \frac{210a}{195} = \frac{14a}{13}$, $c = \frac{210a}{182} = \frac{15a}{13}$. Let

$$s = \frac{1}{2}(a+b+c) = \frac{1}{2}(a + \frac{14}{13}a + \frac{15}{13}a) = \frac{21}{13}a$$

$$A = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{\frac{21}{13}a\left(\frac{21}{13}a - a\right)\left(\frac{21}{13}a - \frac{14}{13}a\right)\left(\frac{21}{13}a - \frac{15}{13}a\right)} = \frac{84}{169}a^2$$

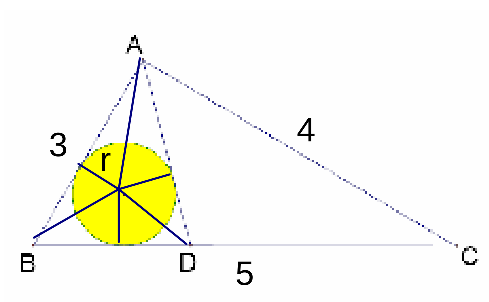
Using $A = \frac{1}{2}a \cdot 210 = \frac{84}{169}a^2$, we have $a = 211.25$. Therefore, $A = 22181.25$.

43) In a right triangle ABC with legs AB=3 and AC=4 the angle bisector AD (D on BC) of the angle A is constructed. Find the radius of the circle inscribed to the triangle ABD. (See the accompanying figure.)

A) $\frac{9-3\sqrt{2}}{7}$ B) $\frac{7-3\sqrt{2}}{9}$ C) $\frac{9-2\sqrt{3}}{7}$

D) $\frac{6-3\sqrt{2}}{5}$ E) $\frac{\sqrt{2}}{9}$

A) Since the triangle ABC is a right triangle, the side $BC = 5$. $\sin \angle BCA = \frac{3}{5}$, $\cos \angle BCA = \frac{4}{5}$, $\sin \angle ABC = \frac{4}{5}$ and $\cos \angle ABC = \frac{3}{5}$. By the law of sine,



$$AD = \frac{4\sin\angle BCA}{\sin\angle ADC} = \frac{4(3/5)}{\sin(180^\circ - 45^\circ - \angle BCA)} = \frac{4(3/5)}{\sin(45^\circ + \angle BCA)}$$

$$= \frac{4(3/5)}{\sin 45^\circ \cos\angle BCA + \cos 45^\circ \sin\angle BCA} = \frac{4(3/5)}{\frac{\sqrt{2}}{2}\frac{4}{5} + \frac{\sqrt{2}}{2}\frac{3}{5}} = \frac{12\sqrt{2}}{7}.$$

Similarly, $BD = \frac{3\sin\angle BAC}{\sin\angle BDA} = \frac{3\sin 45^\circ}{\sin(180^\circ - 45^\circ - \angle ABC)} = \frac{15}{7}$

Note that the radius of the circle is the height of three triangles inside the triangle ABD. Let $s = \frac{1}{2}\left(3 + \frac{12\sqrt{2}}{7} + \frac{15}{7}\right)$. The area of the triangle ABD is

$$A = \sqrt{s(s-3)(s - \frac{12\sqrt{2}}{7})(s - \frac{15}{7})} = \frac{1}{2}3r + \frac{1}{2}\left(\frac{12\sqrt{2}}{7}\right)r + \frac{1}{2}\left(\frac{15}{7}\right)r$$

Therefore, $r = \frac{9 - 3\sqrt{2}}{7}$.

Another quick way to do this problem is to consider the half angles of $\angle BAC$ and $\angle ABC$.

44) Let $r_1, r_2, \ldots, r_n$ be n positive integers, not necessarily distinct, such that

$$(x + r_1)(x + r_2)\cdots(x + r_n) = x^n + 56x^{n-1} + \cdots + 2009.$$

What is the value of n?

A) 4 B) 20 C) 73 D) 758 E) 2009

A) $r_1 + r_2 + \cdots + r_n = 56$ and $r_1 r_2 \cdots r_n = 2009 = 7^2 41$. So none of r_i can be 49 or bigger factor of 2009, otherwise their sum is bigger than 56. One r_i is 41, two r_i's are 7, and the remaining r_i's are 1. Since $56 = 41 + 7 + 7 + 1$, we have $n = 4$.

45) Let $f(x)$ be a function on $(-\infty, \infty)$ and $f(x+2) = f(x-2)$. If $f(x) = 0$ has only three real roots in $[0, 4]$ and one of them is 4, find the number of real roots of $f(x)$ in $(-8, 10]$.

A) 6 B) 7 C) 8 D) 9 E) 10

D) Since $f(4) = 0$, $f(4) = f(2+2) = f(2-2) = f(0) = 0$. $x = 0$ is one root. Let x_0 be another root in $[0, 4]$. Then

$f(4 - x_0) = f(2 + (2 - x_0)) = f(2 - (2 - x_0)) = f(x_0) = 0$. Since $x_0 \neq 0, 4$, we have

$4-x_0=x_0$. So $x_0=2$. On the other hand,
$f(x+4)=f((x+2)+2)=f((x+2)-2)=f(x)$. Thus $f(x)$ is periodic function with the period 4. Therefore, the roots are $-6,-4,-2,0,2,4,6,8,10$. There are 9 roots.

46) In the table below we write all the different products of two distinct counting numbers between 1 and 100:

$$\begin{array}{ccccc} 1\cdot 2, & 1\cdot 3 & \dots & 1\cdot 99, & 1\cdot 100 \\ & 2\cdot 3 & \dots & 2\cdot 99, & 2\cdot 100 \\ & & \ddots & \vdots & \vdots \\ & & & & 99\cdot 100 \end{array}$$

Find the sum of all these products.

A) 300,000 B) 450,000 C) 640,120 D) 12,582,075 E) 25,164,150

D)

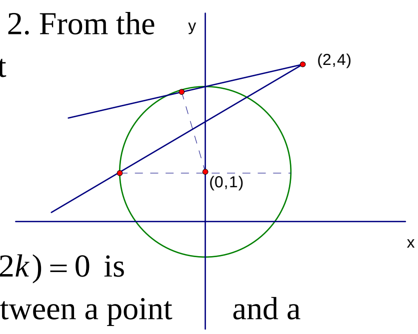
$$\left(\sum_{i=1}^{100} i\right)\left(\sum_{j=1}^{100} j\right)=\sum_{i=1}^{100} i^2+\sum_{i,j=1,i\neq j}^{100} i\cdot j=\sum_{i=1}^{100} i^2+2\sum_{i,j=1,i<j}^{100} i\cdot j$$

$$\sum_{i=1}^{100} i=\frac{(1+100)(100)}{2}=5050 \text{ and } \sum_{i=1}^{100} i^2=\frac{(100)(100+1)(2\cdot 100+1)}{6}=338350$$

Therefore $\sum_{i,j=1,i<j}^{100} i\cdot j=\frac{1}{2}\left(5050^2-338350\right)=12582075$.

47) Suppose that the half circle $y=1+\sqrt{4-x^2}$ and the line $y=kx+4-2k$ have two points of intersection. What is the range of the value k?

A) $[1,3]$ B) $(1,3)$ C) $\left(\frac{1}{2},\frac{3}{4}\right)$ D) $\left[\frac{5}{12},\frac{3}{4}\right]$ E) $\left(\frac{5}{12},\frac{3}{4}\right]$



E) The half circle is centered at $(0,1)$ and has a radius of 2. From the figure on the right, we see that the line intersects at the point $(-2,1)$ on the circle. Thus $1=k(-2)+4-2k$, which gives $k=\frac{3}{4}$. When the line is moving up until it is tangent to the circle, there are two points of intersection. The distance from the center of the circle to the tangent line $kx-y+(4-2k)=0$ is the radius of the circle. Using the formula of the distance between a point and a line, we have

$$\frac{|k\cdot 0-1\cdot 1+4-2k|}{\sqrt{k^2+1}}=2.$$

$k=\dfrac{5}{12}$ is the solution of the equation. Thus when $\dfrac{5}{12}<k\le\dfrac{3}{4}$, there are two points of intersections of the circle and the line.

48) If $x>1$, $y>1$, for what value of x does the equation $x^y=y^x$ in y have only one solution?

A) e B) $\dfrac{e}{2}$ C) 1.6 D) 3 E) 3.5

A) The equation $x^y=y^x$ is equivalent to the equation $y\ln x=x\ln y$ or $\dfrac{\ln y}{y}=\dfrac{\ln x}{x}$ in y for each fixed x. Let function $f(y)=\dfrac{\ln y}{y}$ for $y>1$. Note that $f'(y)=\dfrac{1-\ln y}{y^2}=0$, $y=e$ is only critical number of $f(y)$ and that

$$f''(y)=\frac{-1-2(1-\ln y)}{y^3},\ f''(e)=\frac{-1}{e^3}<0.$$

So $f(y)$ reaches its absolute maximum value at $y=e$ by the second derivative test. Thus the equation $f(y)=f(e)$ has only one solution. Therefore if $\dfrac{\ln x}{x}=f(e)=\dfrac{1}{e}$ or $x=e$, the equation $x^y=y^x$ has only one solution. Note that $f(y)$ is increasing from 0 to $\dfrac{1}{e}$ on $(1,e)$ and decreasing from $\dfrac{1}{e}$ to 0 on (e,∞). For $x\ne e$, $x>1$, the equation has two solutions.

49) Suppose that the function $f(x)=\log_C\dfrac{x-2}{x+2}$ defined for all x in an interval $[a,b]$ is decreasing. For what values of C, there exist a and b ($b>a>2$) such that the range of this function is $[\log_C C(b-1),\log_C C(a-1)]$?

A) $0<C<\dfrac{1}{2}$ B) $0<C<\dfrac{1}{9}$ C) $C>1$ D) $\dfrac{1}{9}<C<1$ E) $\dfrac{1}{3}<C<\dfrac{1}{9}$

B) Since $y=\dfrac{x-2}{x+2}$ is increasing on $(2,\infty)$ and $f(x)$ is decreasing on $[a,b]$, we have $0<C<1$. So the range of $f(x)$ is $\left[\log_C\dfrac{b-2}{b+2},\log_C\dfrac{a-2}{a+2}\right]$. Since the range of $f(x)$ is also $[\log_C C(b-1),\log_C C(a-1)]$. Thus

$$\log_C\frac{b-2}{b+2}=\log_C C(b-1) \text{ and } \log_C\frac{a-2}{a+2}=\log_C C(a-1).$$

This implies that a and b are the solutions of the equation

$\log_C \frac{x-2}{x+2} = \log_C C(x-1)$ or

$Cx^2 + (C-1)x + 2 - 2C = 0$. We know that $x \geq a > 2$. The two different real solutions x_1 and x_2 are greater than 2. From the discriminant

$$\Delta = (C-1)^2 - 4C(2-2C) = 9C^2 - 10C + 1 > 0$$

We have $C < \frac{1}{9}$ and $C > 1$. Since $0 < C < 1$, we conclude that $C < \frac{1}{9}$. Using the fact that $x_1 > 2$ and $x_2 > 2$, we see that $(x_1 - 2) + (x_2 - 2) = x_1 + x_2 - 4 > 0$ and $(x_1 - 2)(x_2 - 2) = x_1x_2 - 2(x_1 + x_2) + 4 > 0$. From the relation between coefficients and roots of a quadratic equation, the following inequalities hold:

$\frac{1-C}{C} > 4$ and $\frac{2-2C}{C} - 2\left(\frac{1-C}{C}\right) + 4 > 0$, which gives $C < \frac{1}{3}$ and $C > 0$. We can conclude that the range of C is $0 < C < \frac{1}{9}$.

50) In a triangle ΔABC, $m\angle ABC = m\angle BCA$ and the points P and Q are located on $\overline{AC}$ and $\overline{AB}$ as shown in the figure so that $AP = PQ = QB = BC$. Find $m\angle BAC$.

A) 15° B) 16° C) 17°

D) 20° E) 21°

D) Let $x = AP = PQ = QB = BC$, $y = AQ$ and $\alpha = \angle BAC$. Note that $AB = AC$. Considering triangles ΔAPQ and ΔABC and by the law of cosine, we have

$$\cos\alpha = \frac{x^2 + y^2 - x^2}{2xy} = \frac{2(x+y)^2 - x^2}{2(x+y)^2}$$

So $\frac{y}{2x} = 1 - \frac{x^2}{2(x+y)^2}$. Let $t = \frac{y}{x}$. Then $\cos\alpha = \frac{t}{2}$ and $\frac{t}{2} = 1 - \frac{1}{2(1+t)^2}$. Or $t^3 = 3t + 1$. Using the identity $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$, we have

$$\cos 3\alpha = 4\left(\frac{t}{2}\right)^3 - 3\left(\frac{t}{2}\right) = \frac{t^3}{2} - \frac{3t}{2} = \frac{3t+1}{2} - \frac{3t}{2} = \frac{1}{2}.$$

Therefore, $3\alpha = 60^\circ$ and $\alpha = 20^\circ$.