Thirty-fourth Annual Columbus State Invitational Mathematics Tournament

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Columbus State University
Department of Mathematics
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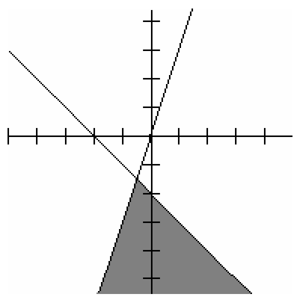
Solutions to the 2008 Mathematics Tournament Problems

1) The answer is B). $\frac{(2y)^{2008}}{2x} \div (x-y)^{2007} = \frac{(-2x)^{2008}}{2x} \div (2x)^{2007} = \frac{(-2x)^{2008}}{(2x)^{2008}} = 1$

2) The answer is D). Draw the lines $y = 3x$ and $y = -2 - 2x$. The solution set for both $y < 3x$ and $y < -2 - x$ is below both lines.



3) The answer is D).

$5^{2008} = 25^{1004} = 25^{1003}(25) = (4 \cdot 6 + 1)^{1003}(25) = (4k+1)(25) = 100k + 25$ for some positive integer k, where we used

$$(4 \cdot 6 + 1)^{1003} = (4 \cdot 6)^{1003} + 1003(4 \cdot 6)^{1002} + \cdots + 1 = 4k + 1.$$

4) The answer is C). Let x be the number correct answers and y be the number of incorrect answers. Then we have the linear system

$$200 + 12x - 3y = 410$$

$$2x - y = 32$$

So $x = 19$ and $y = 6$. There are 50 problems on the test. Therefore, he left 25 problems blank.

5) The answer is B). Let x be the time required to paint one third of the entire room. Then $\left(\frac{1}{40} + \frac{1}{60}\right)x = \frac{1}{3}$. $x = 8$ min.

6) The answer is E). Let x be the number of gallons of brine containing 50% of salt required to make the brine solution. $\frac{10(90\%) + x(50\%)}{10 + x} = 70\%$. $x = 10$.

7) The answer is A). Jane has \$240 and Dick has \$240 at the end. So solving the equation $x-\frac{1}{3}x-\frac{1}{4}(x-\frac{1}{3}x)=240$ gives $x=\$480$.

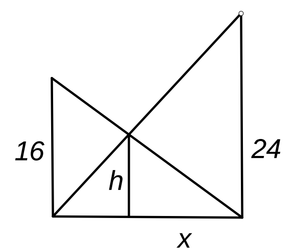
8) The answer is E). Let x be the number of the boys in the class and y be the number of the girls in the class. Then $\frac{3}{4}x+\frac{4}{5}y$ is the number of the students who passed the test and $\frac{3}{4}x=\frac{4}{5}y$. So the fraction of the entire class passed the test is

$$\frac{\frac{3}{4}x+\frac{4}{5}y}{x+y}=\frac{2\left(\frac{4}{5}\right)y}{\frac{16}{15}y+y}=\frac{24}{31}.$$

9) The answer is E). Let x be the mileage. The problem is equivalent to solving the inequality

$$(30)(5)+0.12x<(35)(5)+0.10x. \quad x<1250 \text{ miles.}$$

10) The answer is E). By similarity of triangles, we have $\frac{h}{16}=\frac{x}{13}$ and $\frac{h}{24}=\frac{13-x}{13}$, which gives the equation $\frac{h}{24}=1-\frac{h}{16}$. Therefore, $h=9.6\text{ft}$.



11) The answer is C). ${}_{11}C_5(2x)^6(-3y)^4=-7185024x^6y^5$

12) The answer is B). Let xyz be the three-digit number so that $100x+10y+z=19(x+y+z)$, $100z+10y+x-(100x+10y+z)=297$ and $y-z=3$.

The equations can be reduced to $\begin{cases}9x-y-2z=0\\ z-x=3\\ y=z+3\end{cases}$ Eliminating the variable y yields the system $\begin{cases}3x-z=1\\ -x+z=3\end{cases}$, which yields $x=2$ and $z=5$. Hence $y=z+3=8$.

13) The answer is B). Let x be the width and y be the length. Then

$$120+24=(x+4)(y-3) \text{ and } xy=120.$$

The first equation is equivalent to $148=xy+4y-3x-12$. Replacing xy by 120, we obtain $4y-3x-40=0$. Replacing x by $\frac{120}{y}$ in the previous equation, we have $y^2-10y-90=0$ or $(y-15)(y+6)=0$. Thus $y=15$ and $y=-6$. Since y must be nonnegative, $y = 15$ is the only solution. Then $x = \frac{120}{y} = \frac{120}{15}$ =8. The perimeter is $2(x + y) = 46$ in.

14) The answer is C). $2008_{10} = 11111011000_2$

15) The answer is E). Let $u = \sqrt{x-1}$. Then the equation becomes $\frac{7}{u^2} - \frac{2}{u} + \frac{1}{7} = 0$, which gives $u = 7$. Hence $\sqrt{x-1} = 7$. We have $x = 50$.

16) The answer is D). The ratio of the volume of the scaled model to the volume of actual prism is $1:40^3$. Thus, $\frac{5\text{ cm}^3}{V} = \frac{1}{40^3}$. $V = 320000\text{ cm}^3 = 0.32\text{ m}^3$.

17) The answer is A). If r_1, r_2, and r_3 are three solutions to the equation $x^3 - 27 = 0$, then $x^3 - 27 = (x - r_1)(x - r_2)(x - r_3) = 0$. The sum of the three solutions is the negative coefficient of x^2, which is 0.

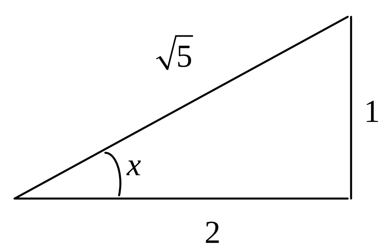
18) The answer is E). The equation $2^{16^x} = 16^{2^x}$ is equivalent to the equation $2^{16^x} = 2^{4(2^x)}$ or $4x = x + 2$. So $x = 2/3$.

19) The answer is B). The equation is equivalent to

$$3(2^x)^2 - 8(2^x) - 3 = 0 \text{ or } (3(2^x) + 1)(2^x - 3) = 0.$$

Solving the equation $2^x = 3$ yields $x = \log_2 3$ and the equation $3(2^x) + 1 = 0$ has no real solutions.

20) The answer is D). $\cos x = 2\sin x$ gives $\cot x = 2$. Using a right triangle with right side lengths 1 and 2, we have $\sin x \cos x = \frac{2}{5}$.



21) The answer is C). $b_1 = 3, b_2 = -2, b_3 = -\frac{1}{3}, b_4 = \frac{1}{2}, b_5 = 3$. So the sequence is periodic with a period 4. Thus $b_1 = b_5 = \cdots = b_{2005} = 3$ and $b_4 = b_5 = \cdots = b_{2008} = \frac{1}{2}$.

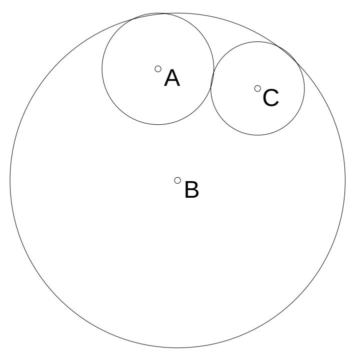
22) The answer is E). You can expect to win 1 time and to lose 9999 times. The probability to win is 1/10000 and the probability to lose is 9999/10000. Therefore the expected value of your gain or loss in dollars is $5000(1/10000) - 1(9999/10000) = -0.4999 \approx -\$0.5$.

23) The answer is A). The probability that you get 4 or 5 correct answers is

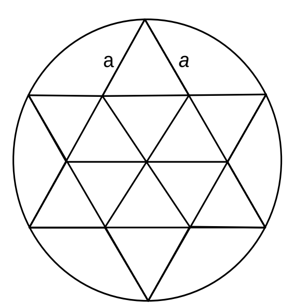
$${}_5C_4(1/5)^4(4/5) + {}_5C_5(1/5)^5 = 0.00672.$$

24) The answer is C). Note that each zero at the end of $2008!$ comes from a factor 10 of the number $2008!$ and that $k \times 2 \times 5^n = k \times 5^{n-1} \times 10$ for natural numbers k and n. So a natural power of 5 and a factor 2 produce a factor 10. There are more even numbers than integer multiple of natural powers of 5 from 1 to 2008. So finding the number of zeros at the end of $2008!$ is equivalent to finding how many integer multiple of the only factors $5, 5^2, 5^3$ and 5^4 in 2008, respectively, since $5^5 = 3125 > 2008$. $2008 \div 5 = 401.6$, $2008 \div 25 = 80.32$, $2008 \div 125 = 16.064$, and $2008 \div 625 = 3.2128$. Therefore, $401 + 80 + 16 + 3 = 500$.

25) The answer is D). Note that
$AB = 21 - 7 = 14, AC = 7 + 6 = 13, BC = 21 - 6 = 15,$ and
by Horen's formula, area $\Delta ABC = \sqrt{s(s-AB)(s-AC)(s-BC)}$=84 , where
$s = (AB + AC + BC)/2.$



26) The answer is A). The letters A, M and T repeat twice in MATHEMATICS, respectively. Other letters repeats only once. The number of distinct permutations using all letters is $\frac{11!}{2!2!2!} = 4989600.$



27) The answer is C). The area of the star is the sum of 12 equilateral triangles with the edge length a. The area of each equilateral triangles with the edge length a is $\frac{\sqrt{3}}{4}a^2$.

Therefore, The area of the star is $12\left(\frac{\sqrt{3}}{4}a^2\right) = 3\sqrt{3}a^2$.

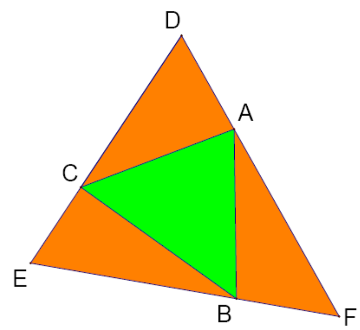
28) The answer is B).

$$\left[\log_a\left(\frac{\log_a a^3}{3a}\right)\right]^{100}=\left[\log_a\left(\frac{3}{3a}\right)\right]^{100}=\left[\log_a a^{-1}\right]^{100}=(-1)^{100}=1$$

29) The answer is A). Let $\theta=\sin^{-1}x$. Then $\sin\theta=x$ and $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$. Since $\cos\theta\ge 0$ for $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, $\cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-x^2}$. Thus $\cot\theta=\frac{\cos\theta}{\sin\theta}=\frac{\sqrt{1-x^2}}{x}$.

30) The answer is E). $Y=104^4+4\times104^3+6\times104^2+4\times104+1=(104+1)^4=105^4$. The prime factorization of 105 is $3\cdot5\cdot7$. So $Y=3^4 5^4 7^4$. All positive factors of Y have the form of $3^n 4^m 7^t$ for n,m,t in $\{0,1,2,3,4\}$. Therefore, the number of positive factors is $5^3=125$.

31) The answer is D).



$$\text{Area }\Delta DEF=\frac{1}{2}(DE)(EF)\sin\angle DEB=\frac{1}{2}(DF)(FE)\sin\angle DFE=\frac{1}{2}(DE)(DF)\sin\angle EDF$$

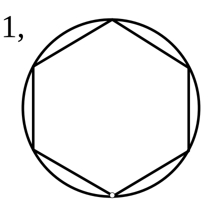
$$\text{Area }\Delta CEB=\frac{1}{2}(CE)(EB)\sin\angle DEB,$$

. $$\text{Area }\Delta BFA=\frac{1}{2}(BF)(FA)\sin\angle DFE,$$

$$\text{Area }\Delta CDA=\frac{1}{2}(DA)(DC)\sin\angle EDF.$$

Using $DC=2EC$, $EB=2BF$, $FA=2AD$, and
$\text{Area }\Delta ABC=\text{Area }\Delta DEF-(\text{Area }\Delta CEB+\text{Area }\Delta BFA+\text{Area }\Delta CDA)$, we have
$(\text{Area }\Delta DEF)/(\text{Area }\Delta ABC)=3$.

32) The answer is B). Consider an inscribed hexagon with side length of 1, which partitions the unit circle into 6 equal arcs. Fix a vertex. The point on the circle having the distance from the vertex greater than or equal to 1 must locate on the four non adjacent arcs. The probability is $\frac{4}{6}=\frac{2}{3}$.



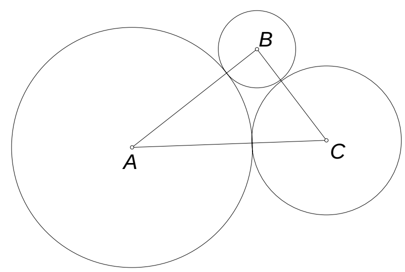
33) The answer is A). Let $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The number of choices for selection of two integers greater than 3 from A is ${}_6C_2$ and The number of choices for selection of two integers smaller than 3 from A is ${}_3C_2$. The probability of the event is $\frac{({}_3C_2)({}_6C_2)}{{}_{10}C_5} = \frac{5}{28}$

34) The answer is C). There are 8 distinct letters and the letters A, M, and T repeat twice. There are ${}_8P_3$ permutations if we choose three distinct letters from 8 distinct letters and there are ${}_3C_1$ choices of picking two same letters and ${}_7C_1$ choices of picking one from other 7 letters, and each choice has three distinct permutations (Ex. MMT, TMM, MTM). So

$$ {}_8P_3 + (3)({}_3C_1)({}_7C_1) = 399 . $$

35) The answer is D) Since distances between the centers of two circles are 3, 4, and 5, which are Pythagorean numbers, the triangle ΔABC is a right triangle with $\angle B = 90^\circ = \frac{\pi}{2}$. The area of ΔABC is

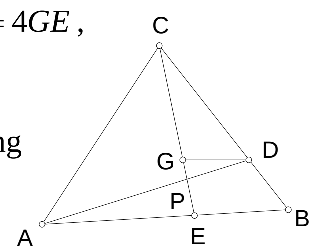
$$\frac{1}{2}(AB)(AC)\sin\angle A = \frac{1}{2}(4)(5)\frac{3}{5} = 6 .$$ Note that



$\sin\angle A = \frac{3}{5}$, $\sin\angle C = \frac{4}{5}$,
$\angle A = 0.6435$ radian and $\angle C = 0.9273$ radian. Subtracting the areas of the three circular sectors from the area of the triangle ABC, we get roughly 0.46 in^2 of the area of the white region between the circles.

36) The answer is E).

Draw $DG \parallel AB$. $\frac{CG}{GE} = \frac{CD}{DB} = \frac{4}{1}, \frac{GD}{EB} = \frac{CD}{CB} = \frac{4}{5}$. Therefore, $CG = 4GE$,

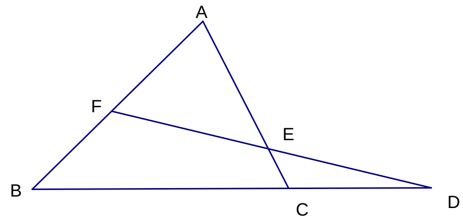


$CG = \frac{4}{5}CE$, and $GD = \frac{4}{5}EB$. Note that $\frac{GD}{AE} = \frac{\frac{4}{5}EB}{AE} = \frac{GP}{PE}$. Using $\frac{EB}{AE} = \frac{3}{4}$, we have $\frac{4}{5}\frac{3}{4} = \frac{GP}{PE}$ and $GP = \frac{3}{5}PE$.

$$CP = CG + GP = \frac{4}{5}CE + \frac{3}{5}PE = \frac{4}{5}(CP + PE) + \frac{3}{5}PE .$$ Thus $\frac{CP}{PE} = 7$.

Menelaus's theorem states:
If a line meets the sides BC, CA, and AB of a triangle in the points D, E, and F then the products of the ratios

$$\frac{AE}{EC}\cdot\frac{CD}{DB}\cdot\frac{BF}{FA}=1$$



By Menelaus's theorem, $\frac{CP}{PE}\cdot\frac{EA}{AB}\cdot\frac{BD}{DC}=1$. Since $\frac{CD}{DB}=\frac{4}{1}$ and $\frac{AE}{EB}=\frac{4}{3}$, we have $\frac{BD}{DC}=\frac{1}{4}$ and $\frac{EA}{AB}=\frac{4}{7}$. Then $\frac{CP}{PE}\cdot\frac{EA}{AB}\cdot\frac{BD}{DC}=\frac{CP}{PE}\cdot\frac{4}{7}\cdot\frac{1}{4}=1$. Therefore, $\frac{CP}{PE}=7$.

37) The answer is C). If $x+\frac{1}{x}=2\cos\frac{\pi}{k}$, then the equation $x^2-2\cos\frac{\pi}{k}x+1=0$ has solutions $x=\cos\frac{\pi}{k}\pm\sin\frac{\pi}{k}i$. So $x=e^{\frac{i\pi}{k}}$ or $e^{-\frac{i\pi}{k}}$.

$$x^k+x^{-k}=(e^{\frac{i\pi}{k}})^k+(e^{\frac{i\pi}{k}})^{-k}=e^{i\pi}+e^{-i\pi}=2\cos\pi=-2.$$

38) The answer is B). $(\sin x+\sin y)^2=\sin^2 x+2\sin x\sin y+\sin^2 y=\frac{1}{2}$ and

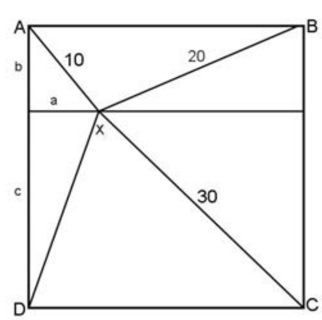
$$(\cos x+\cos y)^2=\cos^2 x+2\cos x\cos y+\cos^2 y=2.$$

Thus

$$\sin^2 x+\cos^2 x+\sin^2 y+\cos^2 y+2(\sin x\sin y+\cos x\cos y)=2+2\cos(x-y)=\frac{1}{2}+2.$$

Therefore, $\cos(x-y)=\frac{1}{4}$.

39) The answer is E). Draw a line passing through the point X and parallel to AB. Label a, b and c as shown in the figure. Then $a^2+b^2=10^2$ and $30^2-c^2=20^2-b^2$, which gives $a^2+c^2+b^2-30^2=10^2-20^2+b^2$. $XD^2=a^2+c^2=30^2+10^2-20^2=600$. We have $XD=10\sqrt{6}$.



40) The answer is C). $2008 = \frac{(a+b)(b-a+1)}{2}$. Since the prime factorization of 2008 is 251×2^3, we have $251\times 16 = (a+b)(b-a+1)$, or $502\times 8 = (a+b)(b-a+1)$, or $1004\times 2 = (a+b)(b-a+1)$. Using the fact that $a>0$ and b is an integer, we see that only $\begin{cases} a+b=251 \\ b-a+1=16 \end{cases}$ has a correct solution. So $a+b=251$.

41) The answer is A). The polynomial $x^4-x^3-2x^2+3x-3$ can be factored as a product of two quadratic polynomials. Since the sum of two solutions of the equation $x^4-x^3-2x^2+3x-3=0$ is 0, the coefficient of x in one of two quadratic polynomials must be zero. Thus

$x^4-x^3-2x^2+3x-3=(x^2+d)(x^2+bx+c)=x^4+bx^3+(c+d)x^2+bdx+cd$.

Then $b=-1$, $bd=-d=3$, and $cd=-3c=-3$. We have

$x^4-x^3-2x^2+3x-3=(x^2-3)(x^2-x+1)$, which gives other two solutions $\frac{1\pm i\sqrt{3}}{2}$.

42) The answer is A). Consider the polynomial $p(x)=(x+1)f(x)-1$. Since each of the numbers $1,2,\ldots,2008$ is a root of $p(x)$ and the degree of $p(x)$ is 2008,

$$p(x)=(x+1)f(x)-1=a(x-1)(x-2)\cdots(x-2008)$$

for some constant a. Substituting $x=-1$ in the above equation, we obtain $-1=a(2009!)$ and so $a=-\frac{1}{2009!}$. Now we evaluate $p(x)$ at -2. We get

$p(-2)=(-2+1)f(-2)-1=-\frac{1}{2009!}(-3)\cdot(-4)\cdots(-2010)=-1005.$ Thus,

$f(-2)=1004$.

43) The answer is A). 2008 has the prime factorization $(251)(2^3)$. Since $aabb$ is the representation of 2008 in base c,

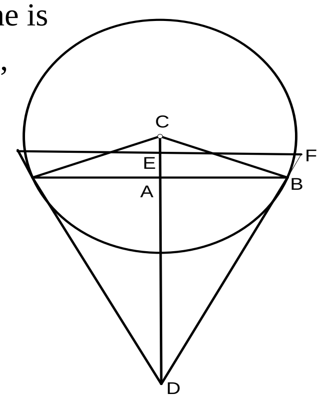
$$2008=ac^3+ac^2+bc+b=ac^2(c+1)+b(c+1)=(ac^2+b)(c+1).$$

So $(ac^2+b)(c+1)=(251)(2^3)$. Note that $ac^2+b>c+1$ because $c\ge 2$, $a\ge 1$, and $b\ge 1$. ac^2+b must be the bigger factor of 2008. If $ac^2+b=251$ and $c+1=8$, then $c=7$ and $49a+b=251$. Using the fact that $0<a,b<c=7$, we see that $a=5$ and $b=6$ are the only solution of the equation $49a+b=251$. Since $5(7^3)+5(7^2)+6(7)+6=2008$, $a-2b+c=5-2(6)+7=0$ is a solution. Other possible factors in $(ac^2+b)(c+1)$ are $ac^2+b=502$ and $c+1=4$ or $ac^2+b=1004$ and $c+1=2$. But in both cases, there are no solutions for a,b and c because $0<a,b<c$ and $c\le 3$.

44) The answer is B). If $\sqrt{a+x} > x$, then $a+\sqrt{a+x} > a+x$. So $\sqrt{a+\sqrt{a+x}} > x$. The equation $\sqrt{a+\sqrt{a+x}} = x$ had no solutions. Similarly, if $\sqrt{a+x} < x$, The equation $\sqrt{a+\sqrt{a+x}} = x$ also had no solutions. If $\sqrt{a+x} = x$, then the equation has a solution. Note that the quadratic equation $x^2 - x - a = 0$ has one negative and one negative solution. The positive solution is the only solution to the equation $\sqrt{a+\sqrt{a+x}} = x$.

45) The answer is D). Note that $n^2 + 2008 = n^2 - 2007^2 + 2007^2 + 2008$. If $n+2007$ divides $n^2 + 2008$, then $n+2007$ must divide $2007^2 + 2008$. The greatest integer n with the property that $n+2007$ divides $2007^2 + 2008$ must satisfy the equation $n + 2007 = 2007^2 + 2008$. Thus $n = 2007^2 + 1 = 4028050$. The sum of the digits of such an integer n is 19.

46) The answer is D). The cross section of the sphere and the cone is shown on the right. Using the fact that $EF = 1$ and $ED = \sqrt{3}$, we have $\angle BDA = 30^\circ$. Since the line segment $\overline{DF}$ is tangent to the circle, $\angle BCD = 60^\circ$. The radius of the sphere is 1. Thus the radius of the intersection of the sphere and the cone is $AB = \sin 60^\circ = \dfrac{\sqrt{3}}{2}$. Therefore, the diameter of the intersection is $\sqrt{3}$.

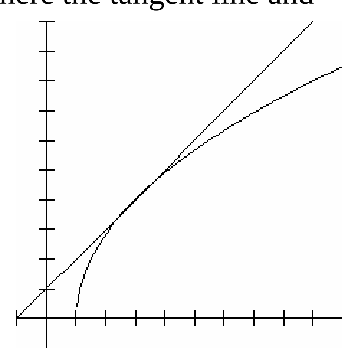


47) The answer is C). Using the fact that the series $\sum_{n=1}^{\infty} x^n$ converges absolutely for $0 < x < 1$, we have

$$a + 2a^2 + 3a^3 + 4a^4 + \cdots = a(1 + 2a + 3a^2 + 4a^3 + \cdots)$$

$$= a\left(\frac{d}{dx}\sum_{n=1}^{\infty} x^n\right)\Bigg|_{x=a} = a\left(\frac{d}{dx}\frac{x}{1-x}\right)\Bigg|_{x=a} = a\left(\frac{1}{(1-x)^2}\right)\Bigg|_{x=a} = \frac{a}{(1-a)^2}$$

48) The answer is B). The slope of the tangent line $y = x + 1$ is 1. Since the derivative $y'(x)$ is the slope the tangent line at x, $y' = 1$ at the point where the tangent line and the parabola touch each other. Implicitly differentiating gives $1 = 2ayy'$. Since $y' = 1$, $y = \dfrac{1}{2a}$. Noting that $y - 1 = x = 1 + ay^2$, we have $\dfrac{1}{2a} - 1 = 1 + a\left(\dfrac{1}{2a}\right)^2$, which gives $a = \dfrac{1}{8}$.



49) The answer is A). Let x the speed of the cars. Two hours after passing over the first car, the plane has traveled (2)(350)=700 miles and the first car has also traveled $2x$ miles. The distance between two cars at the moment is $700-2x$ miles. The two cars met at the middle place of $700-2x$ miles, which is 2400 miles from the plane. The plane has traveled $2400-\frac{700-2x}{2}$ miles since it passed the second car. So we have the equation $\frac{700-2x}{2}=\frac{2400-\frac{700-2x}{2}}{350}x$, which is equivalent to $x^2+2400x-350^2=0$. $x=50$ mph

50) The answer is D).

Let $a_1=1+1$, $a_2=1+\frac{1}{1+1}$, $a_3=1+\frac{1}{1+\frac{1}{1+1}},\ldots$ The sequence $\{a_n\}$ is defined by

$$a_{n+1}=1+\frac{1}{a_n} \text{ for } n\geq 1 \text{ and } a_1=2.$$

Note that the sequence $\{a_n\}$ is convergent and the limit of the sequence is the nonterminating continued fraction. Let $\lim a_n=a$. Taking limit on both sides of

the equation $a_{n+1}=1+\frac{1}{a_n}$,

we get the equation $a^2-a-1=0$. Since $a_n>0$, $a=\frac{1+\sqrt{5}}{2}$.

Note: the convergence of the sequence can be justified by applying the Fixed-Point Theorem to the function $g(x)=1+\frac{1}{x}$ on $[1.5,2]$.