Thirty-third Annual Columbus State Invitational Mathematics Tournament

Sponsored by
Columbus State University
Department of Mathematics
March 3rd, 2007

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ANSWER KEY

The Mathematics Department at Columbus State University welcomes you to our campus and to this year's tournament. We wish you success on this test and in your future studies.

Instructions

This is a 90-minute, 50-problem, multiple choice exam. There are five possible responses to each question. You should select the one "best" answer for each problem. In some instances this may be the closest approximation rather than an exact answer. You may mark on the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer circle the choice you have made on the test booklet. Carefully transfer your answers to the score sheet. Completely darken the blank corresponding to the letter of your response to each question. Mark your answer boldly with a No. 2 pencil. If you must change an answer, completely erase the previous choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response.

The examination will be scored on the basis of +12 for each correct answer, −3 for each incorrect selection, and 0 for each omitted item. Each student will be given an initial score of +200. Pre-selected problems will be used as tie-breakers for individual awards. These problems, designated with an asterisk (*), in order of consideration are: 17, 18, 25, 28, 38, 39, 43, and 50.

Throughout the exam, $\overline{AB}$ will denote the line segment from point A to point B and AB will denote the length of $\overline{AB}$. Pre-drawn geometric figures are not necessarily drawn to scale. The measure of the angle $\angle ABC$ is denoted by $m\angle ABC$.

Review and check your score sheet carefully. **Your student identification number and your school number must be encoded correctly on your score sheet.** When you complete your test, bring your pencil, scratch paper and answer sheet to the test monitor. Leave the room after you have handed in your answer sheet. Please leave quietly so as not to disturb the other contestants. Do not congregate outside the doors by the testing area. You may keep your copy of the test. Your sponsor will have a copy of solutions to the test problems.

Do not open your test until instructed to do so!

1) What is the 2007th digit (to the right of decimal point) in the decimal expansion of $\dfrac{223}{1111}$?

A) 7 B) 5 C) 3 D) 2 $\boxed{\text{E) } 0}$

2) How many ordered pairs of positive integers are solutions of the equation $x^2 + y^2 = 2007$?

$\boxed{\text{A) } 0}$ B) 1 C) 2 D) 3 E) 4

3) Find all real solutions of the equation $\sqrt{8x+5} - x = 2$.

A) $2+\sqrt{5}$ B) 4.236068 C) 4.236068 and -0.236068

$\boxed{\text{D) } 2+\sqrt{5} \text{ and } 2-\sqrt{5}}$ E) -0.236068

4) If $x < -2$, which of the following expressions is equivalent to $\left|1-\sqrt{1+2x+x^2}\right|$?

A) $x+2$ B) $-x$ C) x $\boxed{\text{D) } -x-2}$ E) $-x+2$

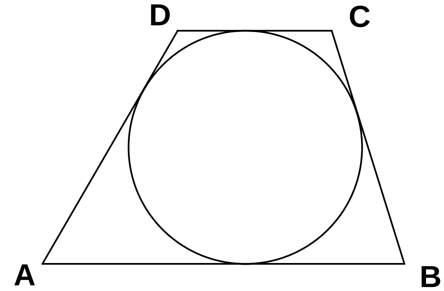
5) In a certain town, 2/3 of the women are married and only 1/2 of men are married. What fraction of the community is single?

A) 1/3 B) 1/2 C) 2/5 D) 4/7 $\boxed{\text{E) } 3/7}$

6) A circle is inscribed in quadrilateral $ABCD$ as shown, with $AB = 17$ and $DC = 12$. If the radius of the circle is 2, what is the area of the quadrilateral?

$\boxed{\text{A) } 58}$ B) 60 C) 78

D) 29 E) 126



7) Find the maximum value of $z = 240x + 160y$ under the assumption that x and y satisfy the following four constraints: $x \geq 0$, $y \geq 0$, $6x + 5y \leq 40$, and $2x + y \leq 10$.

$\boxed{\text{A) } 1400}$ B) 1280 C) 1200 D) 1600 E) 2007

8) Consider the two quadratic equations $x^2 - 2x + p = 0$ and $x^2 - 3x + 4p = 0$. Find p in such a way that these two equations have a common nonzero real solution.

A) $\frac{5}{8}$ $\boxed{\text{B) } \frac{5}{9}}$ C) $\frac{2}{3}$ D) $\frac{4}{9}$ E) $\frac{7}{9}$

9) Fifty-one books are arranged from left to right in order of increasing prices. The price of each book differs by \$1 from that of each adjacent book. The price of the book on the extreme right is five times higher the price of third book. Then:

A) The middle book sells for \$34.

B) The cheapest book sells for \$12

$\boxed{\text{C) The most expensive book sells for \$60}}$

D) The cheapest book sells for \$10.25

E) The most expensive book sells for \$61.25

10) At a fair a vendor has 25 helium balloons on strings: 10 balloons are yellow, 8 are red, and 7 are green. A balloon is selected at random and sold. Given that the balloon sold is yellow, what is the probability that the next balloon selected at random is also yellow?

A) 1/9 B) 9/25 $\boxed{\text{C) } 9/24}$ D) 0 E) 24/25

11) It is claimed that 20% of Americans do not have any health insurance. In a randomly selected group of three people, what is the probability that none of them have health insurance?

A) 0.6 B) 0.3 C) 0.02 D) 0.2 $\boxed{\text{E) } 0.008}$

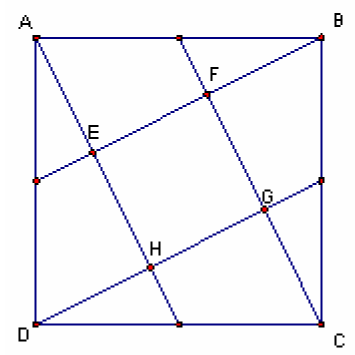
12) A man is walking at a constant speed of 4 miles per hour alongside a railroad track. A freight train, going in the same direction at a constant speed of 30 miles per hour, requires 5 seconds to pass the man. How long, in feet, is the freight train? (1 mile = 5280 feet)

A) 190.67 ft   B) 220 ft   C) 249.33 ft   D) 293.33 ft   E) 185.33 ft

13) A new copy machine can do a certain job in 1 hour less than an older copier. Together they can do this job in 72 minutes. How long would it take the older copier by itself to do the job?

A) 84 min.   B) 240 min.   C) 120 min.   D) 24 min.   E) 180 min.

14) In the figure on the right, the quadrilateral EFGH is created by joining each vertex of square ABCD with the midpoint of an opposite side as indicated in the figure. If $AB = \sqrt{5}$, determine the area of quadrilateral EFGH.



A) $\frac{1}{\sqrt{5}}$ B) $\sqrt{5}$ C) 5

D) 1   E) None of these

15) If a certain operation on one or more members of a set always yields a member of the set, we say that the set is closed under that operation. Then the set $\{n^2 : n \in \mathbb{N}\} = \{1, 4, 9, 16, \cdots\}$ is closed under:

A) Addition   B) Multiplication   C) Geometric average

D) Arithmetic average   E) Subtraction

16) What is the number of ordered pairs (x, y) of positive integers that satisfy the equation $2x + 3y = 563$?

A) 93   B) 94   C) 102   D) 103   E) Infinite

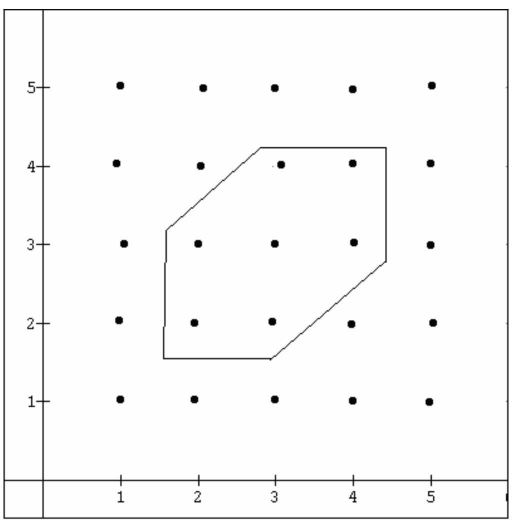
17) * John has an income which is five eighths that of Peter. John's expenses are one-half those of Peter and John saved 40% of his income. What percentage of Peter's income does Peter save?

A) 15% B) 20% C) 23% $\boxed{\text{D) } 25\%}$ E) 31%

18) * What is the value of the sum of all powers of the form $(-1)^n$ if n is an integer such that $0 \le n \le 101$ and n is not a positive multiple of 3?

A) 0 $\boxed{\text{B) } 1}$ C) -1 D) 2 E) -2

19) Let S be the set of all possible ordered pairs you can form using the numbers 1, 2, 3, 4, and 5. The graph of S is given below. In S we define the pair (a,b) to be "less than or equal to" the pair (x,y) if and only if $a \le x$ and $y \le b$. Let A be the subset of S consisting of the seven points enclosed by the irregular hexagon shown in the figure. Find all pairs in S that are less than or equal to all pairs in A.



A) $(1,1), (1,2), (2,1), (2,2)$

B) $(1,1), (1,2), (2,1)$

$\boxed{\text{C) } (1,4), (1,5), (2,4), (2,5)}$

D) $(4,4), (4,5), (5,4), (5,5)$

E) $(4,1), (4,2), (5,1), (5,2)$

20) Find the center C and the radius r of the circle in the *xy*-plane that passes through the points $M = (1,5)$, $N = (2,4)$, and $P = (-3,5)$.

A) $C = (-1,2)$ and $r = 13$ $\boxed{\text{B) } C = (-1,2) \text{ and } r = \sqrt{13}}$ C) $C = (1,-2)$ and $r = \sqrt{13}$

D) $C = (1,2)$ and $r = \sqrt{13}$ E) $C = (-1,2)$ and $r = 169$

21) The lengths of two sides of a triangle are 3 and 10. Which of the following could be the length of the third side?

A) 5 B) 6 $\boxed{\text{C) } 8}$ D) 14 E) 15

22) Which of the following defines the ratio of the surface area to the volume of a right circular cone with radius $r = 1$ and height $h = 1$?

$\boxed{\text{A) } 3+3\sqrt{2}}$ B) $\frac{3\pi+3\sqrt{2}}{\pi}$ C) $\frac{3+3\pi\sqrt{2}}{\pi}$

D) $\frac{\pi+\sqrt{2}}{3\pi}$ E) $\frac{3+3\sqrt{2}}{\pi}$

23) A rhombus is given with one diagonal twice the length of the other diagonal. Express the area of the rhombus in terms of s, where s is the side of the rhombus.

$\boxed{\text{A) } \frac{4}{5}s^2}$ B) $\frac{5}{4}s^2$ C) $\frac{2}{5}s^2$ D) $\frac{5}{2}s^2$ E) s^2

24) The geometric mean of three positive real numbers is 3. How small can their sum be?

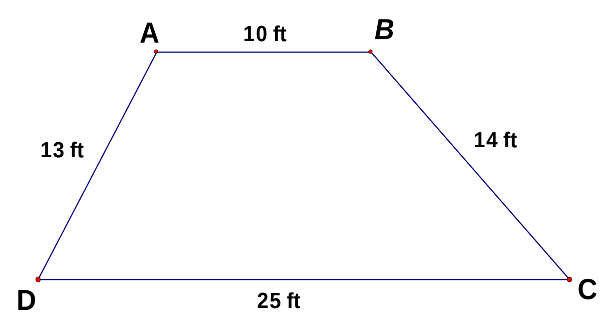
A) 3 B) 1 C) $\frac{1}{3}$ $\boxed{\text{D) } 9}$ E) None of these

25) * The figure (not drawn to scale) shows a trapezoid $ABCD$ with dimensions $AB = 10$ ft, $BC = 14$ ft, $DC = 25$ ft, and $AD = 13$ ft. Find the area of the trapezoid.

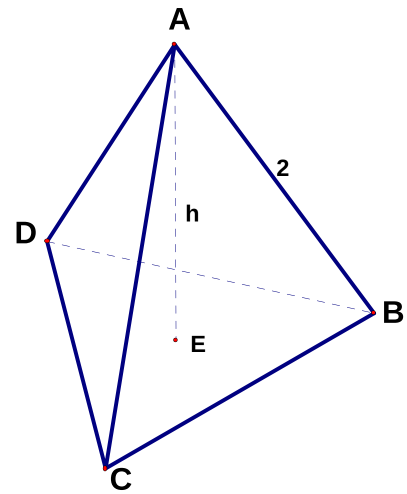
$\boxed{\text{A) } 196 \text{ ft}^2}$ B) 196.44 ft^2

C) 206 ft^2 D) 186 ft^2

E) 187 ft^2



26) All edges of the regular tetrahedron shown in the figure have length 2 feet. Find the height h of the tetrahedron.

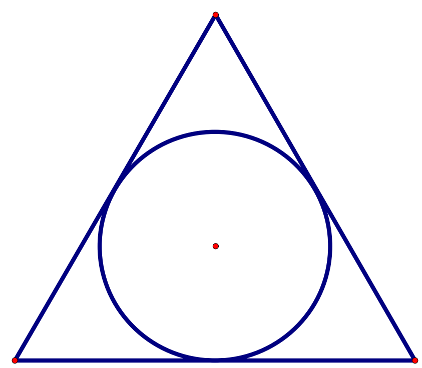


A) $\frac{3\sqrt{6}}{2}$ B) $\frac{\sqrt{6}}{2}$

C) $\boxed{\frac{2\sqrt{6}}{3}}$ D) $\frac{\sqrt{6}}{3}$

E) None of these

27) The length of the side of the equilateral triangle shown in the figure is 2 feet. Find the ratio of the area of the triangle to the area of the inscribed circle.



A) $\boxed{\frac{3\sqrt{3}}{\pi}}$ B) $\frac{\sqrt{3}}{\pi}$ C) $\frac{\sqrt{3}}{3\pi}$

D) $\frac{3}{\pi}$ E) $\frac{3\pi}{\sqrt{3}}$

28) * If a stock rose 10% in the first year, 20% in the second year and fell 15% in the third year, what is its average rate of return (rounded to two decimal places)?

A) 4.69 %   B) 4.06 %   C) $\boxed{\text{3.91 \%}}$ D) 5 %   E) 15 %

29) Find the remainder of 2^{2007} when divided by 7.

A) 0   B) $\boxed{1}$ C) 2   D) 3   E) 4

30) Find all real values of m for which the graph of the line $y = mx$ does not intersect the graph of the equation $9x^2 - 4y^2 = 36$.

A) $-\frac{3}{2} \le m \le \frac{3}{2}$ B) $-\frac{3}{2} < m < \frac{3}{2}$ C) $-\frac{3}{2} \le m \le \frac{\pi}{2}$ or $\frac{3}{2} \le m \le \frac{\pi}{2}$

$\boxed{\text{D) } -\infty < m \le -\frac{3}{2} \text{ or } \frac{3}{2} \le m < \infty}$ E) $-\infty < m < -\frac{3}{2}$ or $\frac{3}{2} < m < \infty$

31) How many ordered triples (a,b,c) of nonnegative integers are solutions of the equation $a+b+c = 2007$?

A) 2,015,018 $\boxed{\text{B) } 2{,}017{,}036}$ C) 2,015,028

D) 2,013,021 E) None of these

32) Which of the following is the value of the sum $\sin 1^\circ + \sin 2^\circ + \sin 3^\circ + \cdots + \sin 360^\circ$?

A) 2π B) $\sin 1^\circ$ C) $-\sin 1^\circ$ D) $\sin 359^\circ$ $\boxed{\text{E) } 0}$

33) Find the area of the region bounded by the graph of the equation $|x+y| + |y| = 4$.

A) 16 $\boxed{\text{B) } 32}$ C) 48 D) 8 E) 64

34) What is the largest irrational solution of the equation $x^4 - 2x^3 - 3x^2 + 2x + 2 = 0$?

A) $-1+\sqrt{3}$ B) $-1-\sqrt{3}$ C) $1-\sqrt{3}$ $\boxed{\text{D) } 1+\sqrt{3}}$

E) The equation has no irrational solutions.

35) Find the coefficient of the term containing x^{-1} in the expansion of $\left(\frac{a^2}{\sqrt{x}} - \frac{\sqrt{x}}{a}\right)^6$.

A) $\frac{15}{x}$ B) $\frac{20a^3}{x}$ $\boxed{\text{C) } 15a^6}$ D) $20a^3$ E) $-20a^3$

36) Let $i=\sqrt{-1}$. What is the sum of $1+2i+3i^2+\cdots+401i^{400}$.

A) $1+i$ B) 201 $\boxed{\text{C) } 201-200i}$ D) $201+200i$

E) $201i$

37) An ordered pair (a,b) is selected at random from the set of pairs $P=\{(x,y)\mid x=0,1,\ldots,8 \text{ and } y=0,1,\ldots,222\}$. What is the probability that for the selected pair we have $a>b$?

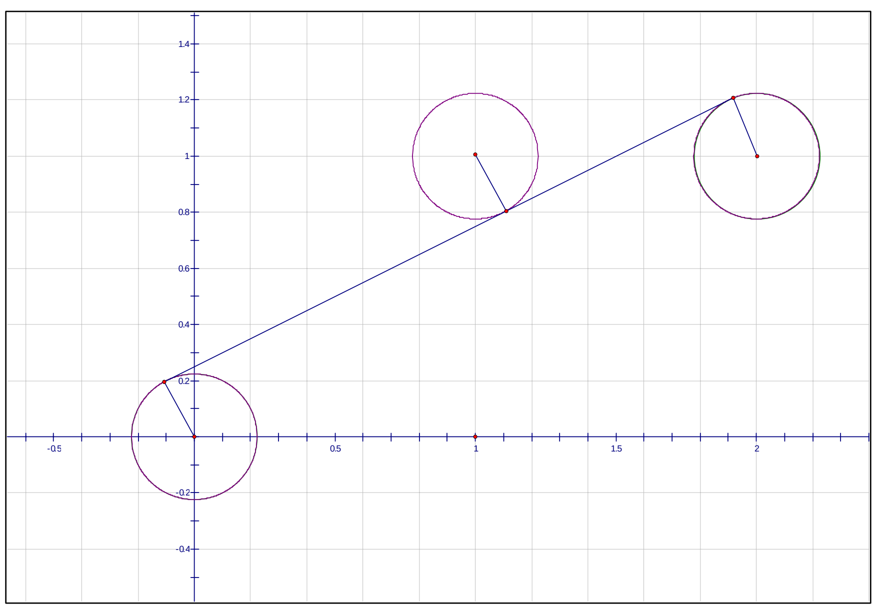
$\boxed{\text{A) } \frac{4}{223}}$ B) $\frac{8}{222}$ C) $\frac{28}{2007}$

D) $\frac{5}{223}$ E) $\frac{9}{223}$

38) * There are two positive solutions to the equation $\log_{2x}2+\log_4 2x=-\frac{3}{2}$. What is the product of the two solutions?

A) $1/21$ B) $3/21$ $\boxed{\text{C) } 1/32}$ D) $1/8$ E) 2

39) * Three circles, each of the same radius r, have centers at $(0,0)$, $(2,1)$, and $(1,1)$. If they have a common tangent line, as shown in the figure, find their common radius r.



A) $\sqrt{5}$ B) 2

$\boxed{\text{C) } \frac{\sqrt{5}}{10}}$ D) $\frac{\sqrt{5}+1}{2}$

E) $\frac{\sqrt{5}+1}{10}$

40) If $0 < b < c$, find the value of $\sin\left(2\cos^{-1}\left(\frac{b}{c}\right)\right)$.

A) $\dfrac{2b}{\sqrt{c^2+b^2}}$ B) $\dfrac{2c}{\sqrt{c^2-b^2}}$ C) $\boxed{\dfrac{2b\sqrt{c^2-b^2}}{c^2}}$

D) $\dfrac{2b}{c^2\sqrt{c^2-b^2}}$ E) $\dfrac{2b}{c\sqrt{c^2-b^2}}$

41) If $f(x) = \dfrac{\tan x - 2}{\sec x}$, then the derivative of f, where it exists, is

A) $f'(x) = \cos x - \sin x$ B) $f'(x) = 2\cos x - \sin x$ C) $f'(x) = 2\cos x + \sin x$

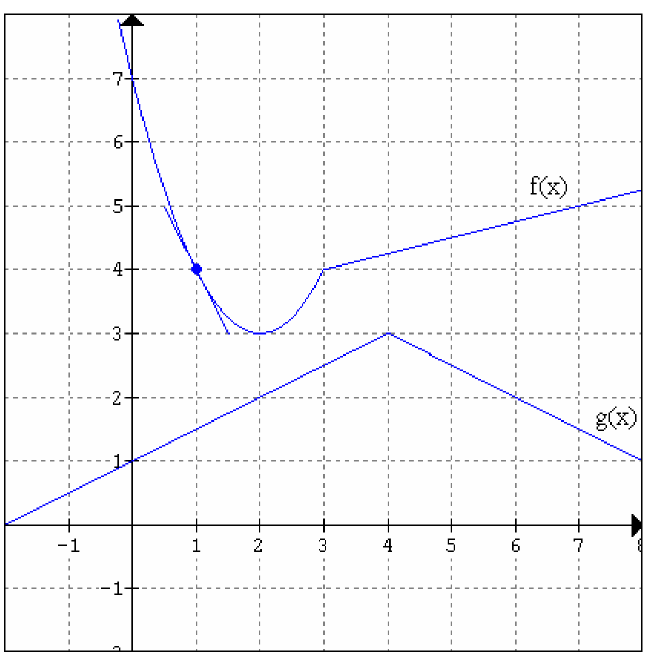
D) $f'(x) = \cos x + \sin x$ E) $\boxed{f'(x) = \cos x + 2\sin x}$

42) Let E be the set of all positive even integers and let S be the set of all positive integers k such that k^2 is even. Which of the following is true?

A) $E \subset S$ and $E \neq S$ B) $S \subseteq E$ and $E \neq S$ C) $S \cap E = \varnothing$

D) $\boxed{S = E}$ E) None of these

43) * If $P(x) = f(g(x))$ where $f(x)$ and $g(x)$ are the functions whose graphs are shown, which of the following is the best estimate for $P'(0)$?



A) $P'(0) = -\dfrac{1}{4}$ B) $P'(0) = -2$

C) $P'(0) = 4$ D) $\boxed{P'(0) = -1}$

E) $P'(0) = 1$

44) Let $y=(x-a)^2+(x-b)^2+(x-c)^2$, where the constants a, b, and c are nonnegative real numbers. For what value of x is y a minimum?

A) $\boxed{\dfrac{a+b+c}{3}}$ B) $a+b+c$ C) $\sqrt{abc}$ D) $\sqrt{\dfrac{a+b+c}{3}}$

E) $\sqrt{\dfrac{a^2+b^2+c^2}{3}}$

45) Given a number of the form $a+b\sqrt{2}$, where a,b are rational numbers with $a\neq 0$ or $b\neq 0$, there exists a number $p+q\sqrt{2}$, with p,q rational numbers, such that $\left(a+b\sqrt{2}\right)\left(p+q\sqrt{2}\right)=1$. What is the value of p^2-2q^2?

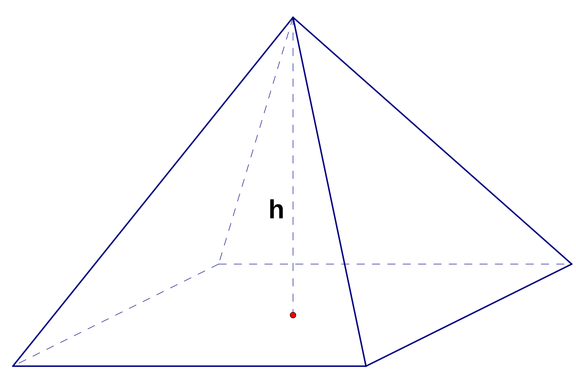
A) $\dfrac{a^2-2b^2}{\left(a^2-4b^2\right)^2}$ B) $\dfrac{a^2-2b^2}{a^2+2b^2}$ C) $\boxed{\dfrac{1}{a^2-2b^2}}$ D) $\dfrac{-1}{a^2+2b^2}$ E) $\dfrac{1}{a^2-4b^2}$

46) The pyramid of square base shown in the figure has a height of h feet. If all edge lengths are the same, what value of h will make the volume of the pyramid equal to 3 cubic feet?

A) $\sqrt{\dfrac{9}{2}}$ ft B) $\boxed{\sqrt[3]{\dfrac{9}{2}}\text{ ft}}$

C) $\sqrt{\dfrac{3}{2}}$ ft D) $\sqrt[3]{\dfrac{3}{2}}$ ft

E) $\sqrt[3]{9}$ ft



47) Solve the equation $2^x+4^y=2^y+4^x$ for y in terms of x and find the sum of the solutions if $x<0$.

A) $x\log_2\left(1-2^x\right)$ B) $\boxed{x+\log_2\left(1-2^x\right)}$ C) $(x+1)\log_2\left(1-2^x\right)$

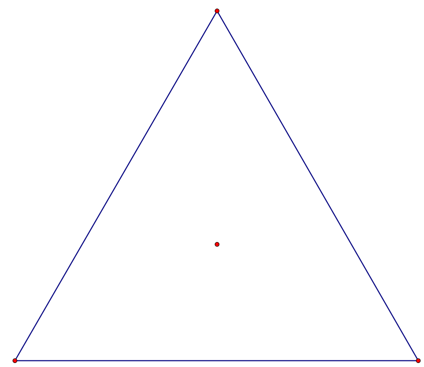
D) $x\log_2\left(1+2^x\right)$ E) $(x+1)+\log_2\left(1-2^x\right)$

48) What is the sum of the coefficients of all the terms containing only the even powers of x in the expansion of $\left(1+2x+3x^2-4x^3\right)^9$?

A) $10,077,184$ B) $-10,077,184$ C) $5,038,592$

$\boxed{\text{D) } 5,039,104}$ E) $10,078,208$

49) A point is selected at random from the interior of an equilateral triangle. Find the probability that the selected point is closer to a vertex than to the center of the triangle.



A) $\frac{\sqrt{3}}{\pi}$ B) $\frac{2}{3}$ C) $\frac{2}{9}$

D) $\frac{1}{9}$ $\boxed{\text{E) } \frac{1}{3}}$

50) * Evaluate $\lim_{n\to\infty}\int_0^1 x^n \sin(nx)\,dx$

A) $\pi/2$ B) $\pi/4$ C) 1 $\boxed{\text{D) } 0}$ E) Does not exist.