Thirty-second Annual Columbus State Invitational Mathematics Tournament

Sponsored by
Columbus State University
Department of Mathematics
March 4th, 2006

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Solutions to the 2006 Mathematics Tournament Problems

1) $x^3 - 78x^2 + 1155x - 2006 = (x-2)(x-59)(x-17)$, therefore the answer is 78

2) The employee would pay 80 % of 70% of \$100, namely \$56

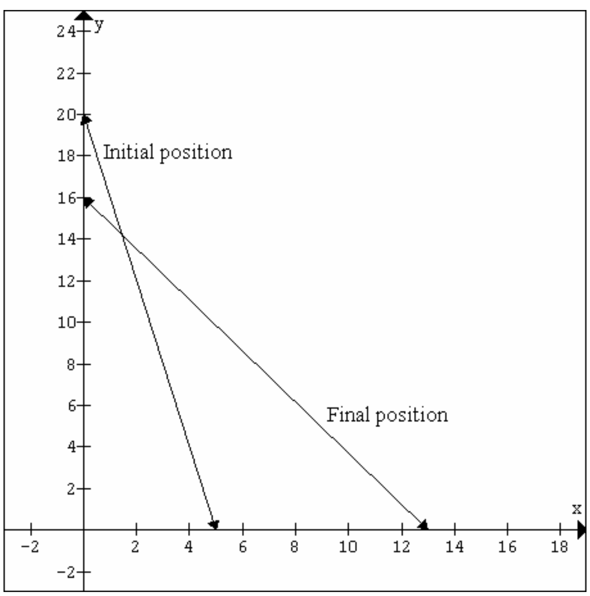
3) The answer is 8^{2^4} since $8^{2^4} = 8^{16} = 2^{48} \neq 2^{2^{2^2}} = 2^{16}$

4) Let x denote the number of people in the audience at the beginning of the lecture. Then

$\frac{3}{4}\left(\frac{2}{3}\left(\frac{1}{2}x\right)\right) = 9$ so the answer is 36

5) Multiply the expression $\frac{1}{x-3} + \frac{1}{x+3} = \frac{6}{x^2-9}$ by $x^2 - 9$ to get $(x+3)+(x-3) = 6$ or $x = 3$. Since this value is not in the domain of the variable for this expression the answer is: The equation has no solutions

6) A From the initial position we can compute the length of the pole to be $\sqrt{20^2 + 5^2} = \sqrt{425}$. Therefore in the final position the base of the pole will be at $\sqrt{\left(\sqrt{425}\right)^2 - 16^2} = \sqrt{169} = 13$ feet.



7) Since $\sqrt{x+4}$ is defined when $x \geq -4$ and $\sqrt{x-1}$ is defined when $x \geq 1$ the identity $\sqrt{x^2 + 3x - 4} = \sqrt{(x+4)(x-1)} = \sqrt{x+4}\sqrt{x-1}$ is true for all real numbers in $[1, \infty)$

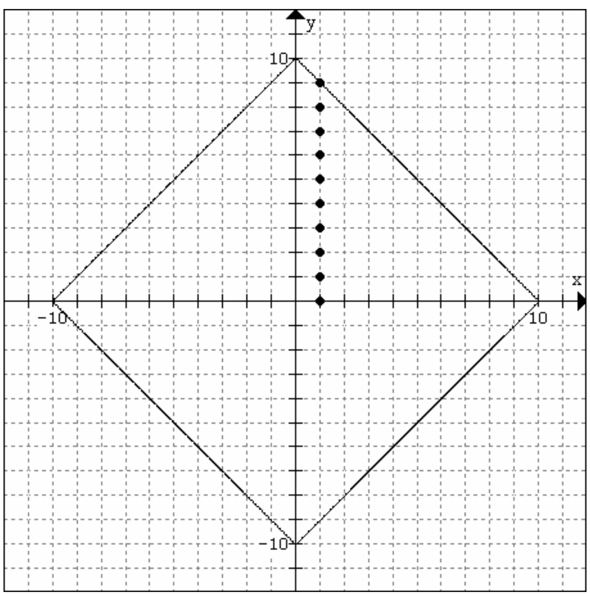
8) The inequality $x^2 - 3x < |4x - 6|$ means that we need to consider the inequalities:

$$\begin{array}{lcl} x^2 - 3x < 4x - 6 & & -x^2 + 3x > 4x - 6 \\ x^2 - 7x + 6 < 0 & \text{or} & 0 > x^2 + x - 6 \\ (x-6)(x-1) < 0 & & 0 > (x+3)(x-2) \end{array}$$

The solution set of the first one is $1 < x < 6$ and the solution set of the second one is $-3 < x < 2$. The union of the two is $-3 < x < 6$.

9) $\sqrt[4]{7x^2 - 6} = x$ implies $7x^2 - 6 = x^4$ which is $x^4 - 7x^2 + 6 = 0$ or $x^4 - 7x^2 + 6 = (x^2 - 6)(x^2 - 1) = 0$ which means that $x = \pm\sqrt{6}$ or $x = \pm 1$. Since we must have $x > 0$ the sum of the solutions of is $1 + \sqrt{6}$.

10) The pattern indicates we should consider the first quadrant with $x \geq 1$ and $y \geq 0$. Then there $1 + 2 + \cdots + 10 = 55$ pairs satisfying such condition; by symmetry there are $4 \times 55 = 220$ pairs with the desired property, but do not forget the center!!!, so there are 221 such points.



11) $|x^2 - 4| \leq 0.25$ means that $-0.25 \leq x^2 - 4 \leq 0.25$ or $4 - 0.25 \leq x^2 \leq 0.25 + 4$ or equivalently, that $\frac{\sqrt{15}}{2} \leq x \leq \frac{\sqrt{17}}{2}$ after taking square roots and noticing that $x \geq 0$. The last inequality is equivalent to $\frac{\sqrt{15}}{2} - 2 \leq x - 2 \leq \frac{\sqrt{17}}{2} - 2$; this means we must take δ to be $\frac{\sqrt{17}}{2} - 2$ because $\frac{\sqrt{17}}{2} - 2 < \left|\frac{\sqrt{15}}{2} - 2\right|$.

Therefore the answer is $\frac{\sqrt{17}}{2} - 2 = 0.06155281281$

12) From the first student we can get the value of the constant b by using the fact that $\frac{b}{2}=(-4)(3)$, so $b=-24$. From the second student we get the value of a by using the fact that $\frac{a}{2}=-(5-4)$ so $a=-2$ and $2x^2+ax+b=2x^2-2x-24=2(x-4)(x+3)=0$. The solution is $\{4,-3\}$

13) If $\log_a x=4$ then $x=a^4=\left(\frac{1}{a}\right)^{-4}$ therefore $\log_{1/a} x=-4$.

14) First notice that a, being a base, must be positive. Since $a^{2+\log_a 4}=a^2a^{\log_a 4}=4a^2$ we have that the original equation is equivalent to $3a^2=2$ or $a=\sqrt{\frac{2}{3}}$

15) The definition of $\lceil x \rceil$ implies that $\lceil \sqrt[3]{x} \rceil=3$ means $2<\sqrt[3]{x}\le 3$ or $8<x\le 27$; similarly, $\lceil \sqrt[3]{y} \rceil=4$ means that $27<y\le 64$ and therefore $35<x+y\le 91$ which means that the smallest possible value of $\lceil x+y \rceil$ is 36.

16) A three-digit palindromic number is of the form aba for $a=1,\cdots,9$ and $b=0,1,\cdots,9$. Therefore there are $9\times 10=90$ such numbers.

17) From the previous problem notice that a three-digit palindromic number is of the form $aba=100a+10b+a$ for $a=1,\cdots,9$ and $b=0,1,\cdots,9$. Their sum is

$$\sum_{b=0}^{9}\sum_{a=1}^{9} aba=\sum_{b=0}^{9}\sum_{a=1}^{9}(100a+10b+a)=\sum_{b=0}^{9}\left(\sum_{a=1}^{9}100a+\sum_{a=1}^{9}10b+\sum_{a=1}^{9}a\right)=\sum_{b=0}^{9}\left(100\times\frac{9(9+1)}{2}+10\times 9b+\frac{9(9+1)}{2}\right)=$$

$$\sum_{b=0}^{9}(4500+90b+45)=4500\times 10+90\times\frac{9(9+1)}{2}+45\times 10=49500$$

18) If the first (top one) layer has 6 logs then the second one has 6+2 logs , the third one 6+4 logs , etc,. This means that the i-th layer has $6+2(i-1)$ logs, $i=1,\cdots,n$. Since the sum of logs in all layers is $4n+110$, we have $\sum_{i=1}^{n}(6+2(i-1))=4n+110$, that is $\sum_{i=1}^{n}6+2\sum_{i=1}^{n}i-\sum_{i=1}^{n}2=4n+110$ or $6n+2\frac{n(n+1)}{2}-2n=4n+110$ which leads to $(n+11)(n-10)=0$ and to the answer $n=10$.

19) If both n^2+2 and $n+1$ are divisible by k then so is $n^2+2-(n+1)(n-1)=3$. That is, if n^2+2 and $n+1$ have a common factor greater than 1 then it is 3. If $n+1$ is divisible by 3 then $n=3k'-1$ or equivalently $n=3k+2$. Also $n^2+2=(3k+2)^2+2=3(3k^2+k+2)$ is divisible by 3. Since $n=3k+2$ must be an integer between 1 and 100 we have that k must be an integer between 0 and 32. Therefore there 33 such integers n.

20) $\sum_{i=1}^{n}\frac{2006}{i(i+1)}=2006\sum_{i=1}^{n}\frac{1}{i(i+1)}=2006\sum_{i=1}^{n}\left(\frac{1}{i}-\frac{1}{(i+1)}\right)=2006\left(1-\frac{1}{n+1}\right)=2006\frac{n}{n+1}$ which means that $2006\frac{n}{n+1}=2005$ when $n=2005$

21) Notice that $2^{2006}+1=\left(2^{2006}-1\right)+2=3k+2$ for some integer k. This implies that the remainder of $2^{2006}+1$ when divided by 3 is 2. (notice that $x^{2006}-1=\left(x^2\right)^{1003}-1=(x^2-1)q(x^2)$ for some polynomial $q(x^2)$ and then let $x=2$)

22) If we let $t=\frac{x}{x-1}$ the original expression becomes $t^2=-6t+7$ or $t^2+6t-7=(t+7)(t-1)=0$. This means that $-7=\frac{x}{x-1}$ which has one solution or $1=\frac{x}{x-1}$ which does not have any solutions. The only solution is then $x=\frac{7}{8}$

23) Let $t=0$ be the time at sunrise and T the time elapsed until they met at noon. If we measure from point A the distance travel by each driver, then we have $d_a=v_a t$ and $d_b=D-v_b t$ where D is the distance from A to B. When they both met $d_a=d_b$, that is $D-v_bT=v_aT$ or $\boxed{D=v_aT+v_bT}$. Four hours later driver A made to point B, that is $\boxed{D=v_a(T+4)}$ and while nine hours later driver B made it to point A, that is, $0=D-v_b(T+9)$ or $\boxed{D=v_b(T+9)}$ using the three equations in the boxes to eliminate D we get that $T=\frac{v_a}{v_b}4$ and $T=\frac{v_b}{v_a}9$ which implies $T=6$. We conclude that sunrise was at 6 am.

24) If r, b denote the number of red and blue chips, respectively, then the information provided means that we have three equations: $r+b=16$, $\frac{r}{16}\frac{b}{15}=\frac{1}{4}$, and $\frac{b}{16}\frac{b-1}{15}=\frac{1}{8}$. Solving these equations lead to the conclusion that $r=10$.

25) To go to destination A the passenger must arrive at a time t satisfying $7:05<t\le 7:15$, or $7:20<t\le 7:30$, or $7:35<t\le 7:45$, or $7:50<t\le 8:00$. Therefore the passenger will go to destination A $\frac{40}{60}=\frac{2}{3}$ of the times.

26) The number of plates with exactly one H and exactly two 4's is

$$3\times(1\times 25\times 25)\times 6(1\times 1\times 9\times 9)=911250$$

The number of plates with exactly two H's and exactly two 4's is

$$3\times(1\times 1\times 25)\times 6(1\times 1\times 9\times 9)=36450$$

The number of plates with exactly three H's and exactly two 4's is

$$(1\times 1\times 1)\times 6(1\times 1\times 9\times 9)=486$$

Therefore the answer is 948,186.

27) The sum is

$$(1+2+\cdots+100+101)-(3+6+\cdots+99)=\frac{101\times 102}{2}-3(1+2+\cdots+33)=\frac{101\times 102}{2}-3\frac{33\times 34}{2}=3468$$

28) To find x if $3^{27^{(669x-1)}}=27^{3^{2005x}}$ notice the right hand side can be written as $27^{3^{2005x}}=(3^3)^{3^{2005x}}=3^{3\times 3^{2005x}}=3^{3^{2005x+1}}$ and that the left hand side is $3^{27^{(669x-1)}}=3^{3^{3(669x-1)}}$. This means that we must have $3(669x-1)=2005x+1$ or $2007x-3=2005x+1$. That is, $x=2$.

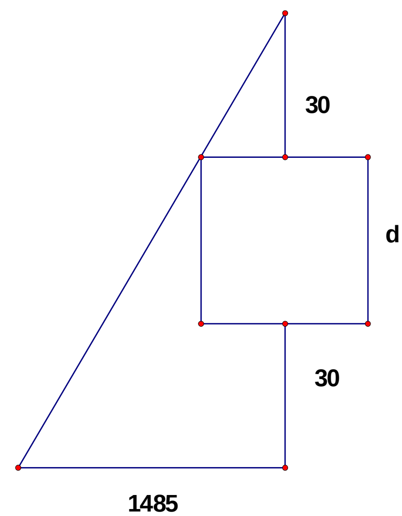
29) What is the number of integers between 1 and 2006 (inclusive) that are not divisible by 2 or 5? There 1003 integers divisible by 2; there $\frac{2006}{5}=401$ (rounded down) integers divisible by 5 and there 200 integers divisible by 10. Therefore the number of integers that are not divisible by 2 or 5 is equal to

$$2006-(1003+401-200)=802$$

30) Three distinct lines will intersect (in the way described in the problem) in at most 3 points; if we add a circle the intersection points could increase by at most 6 points; by adding one more (distinct) circle we could add at most 6 points of intersection with the three lines and at most 2 with the other circle. Therefore the answer is 17.

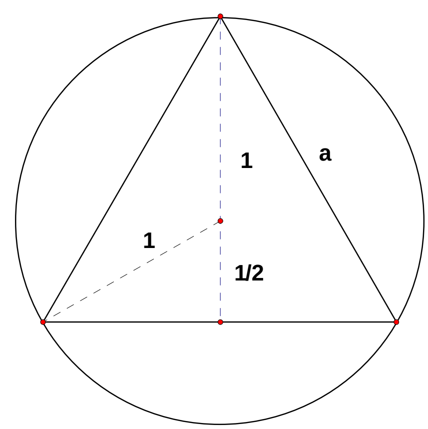
31) From the figure we get that $\frac{30}{d/2} = \frac{30 + d + 30}{1485}$ by similar triangles.

This leads to the equation $d^2 + 60d - 89100 = (d - 270)(d + 330) = 0$ or $d = 270$.



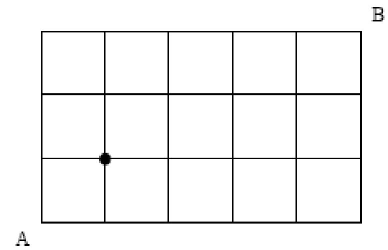
32) If the length of a side of the triangle is a then

$\frac{a}{2} = \sqrt{1^2 - (1/2)^2} = \frac{\sqrt{3}}{2}$. Therefore the area of the triangle is $\frac{3\sqrt{3}}{4}$



33) Clearly, the grazing area is $\frac{3}{4}\pi 60^2 + \frac{1}{2}\pi 20^2 = 2900\pi$.

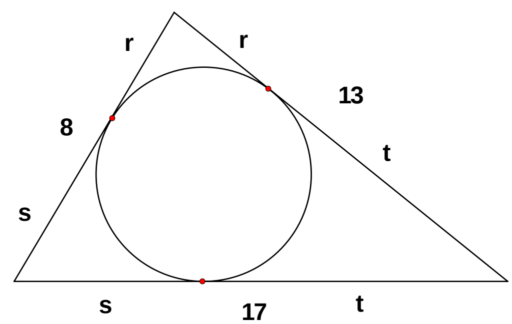
34) If we denote by R a motion to the right and by U a motion in the upward direction a generic route from A to B will consists (If we ignore the fact that the node • must be avoided) of eight letters chosen from the two letters R or U. For instance, one route could be RURRRURU. Notice that we must always have five R's and three U's. In order to count all different routes from A to B we must count all possible arrangements of five R's and three U's. If we are to avoid the node then the routes must have one of the following two forms RR_ _ _ _ _ _ or UU_ _ _ _ _ _, where the remaining 6 spaces are filled with 3 R's and 3 U's in the first case or with 5 R's and 1 U in the second case. The numbers of routes is then $\frac{6!}{3!3!}+\frac{6!}{5!1!}=20+6=26$.



35) From the picture we get the equations

$$r+s=8$$
$$r+t=13$$
$$s+t=17$$

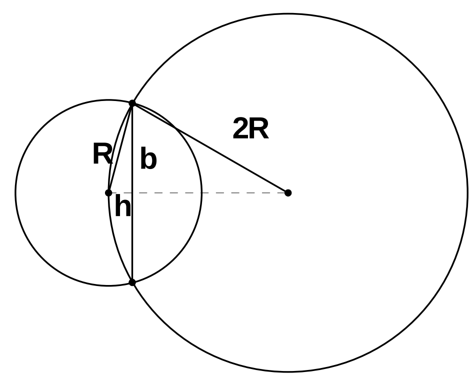
that can be solved to yield $\frac{r}{s}=\frac{1}{3}$



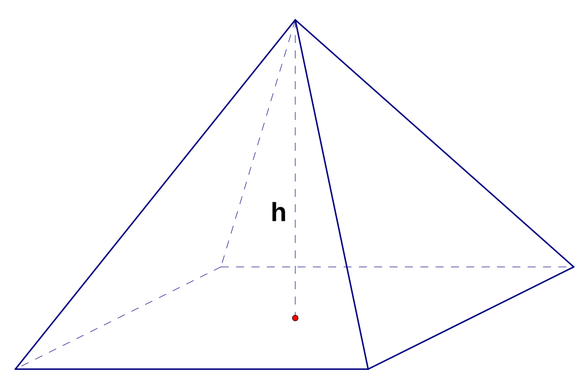
36) The length is $2b$. To find b notice that from the figure we can write the equations.

$$(2R)^2=b^2+(2R-h)^2$$
$$R^2=b^2+h^2$$

that lead to $2b=\frac{\sqrt{15}}{2}R$



37) If we let a be the length of any edge, then half the diagonal of the base is $\frac{\sqrt{2}}{2}a$. Then using the right triangle formed by the top vertex, the center of the base and any of the vertices of the base we can write the equation $a^2=h^2+\left(\frac{\sqrt{2}}{2}a\right)^2=1+\frac{1}{2}a^2$ or $a^2=2$ which means $V=\frac{2}{3}$.

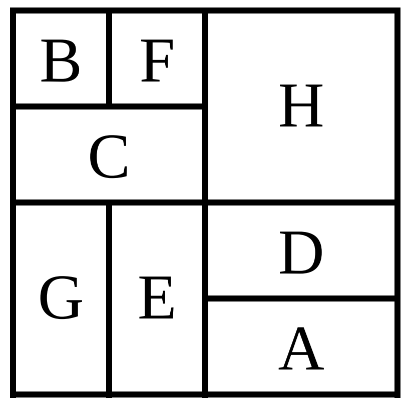


38) You are about to leave for school in the morning and discover you do not have your glasses. You know the following statements are true.

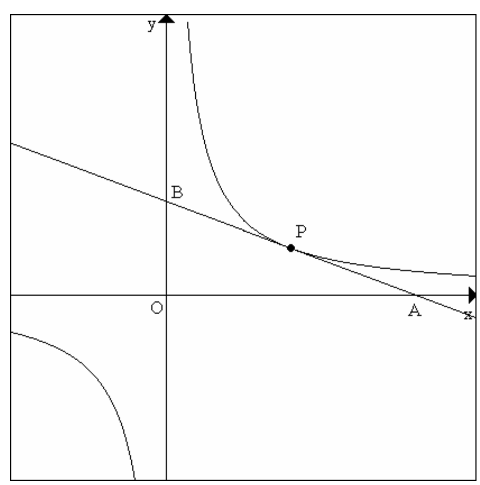
I. If my glasses are on the kitchen table, then I saw them at breakfast.
II. I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.
III. If I was reading the newspaper in the living room, then my glasses are on the coffee table.
IV. I did not see my glasses at breakfast.
V. If I was reading my book in bed, then my glasses are on the bed table.
VI. If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.

Notice that IV and I imply that “my glasses are on the kitchen table” is false. Therefore from VI we conclude that “I was reading the newspaper in the kitchen” must also be false; this implies that from II the statement “I was reading the newspaper in the living room “ must be true; using this in III we conclude that “my glasses are on the coffee table” must be true. (Notice that V is not needed).

39) Clearly the order is B, F, C, G, E, A, D, H



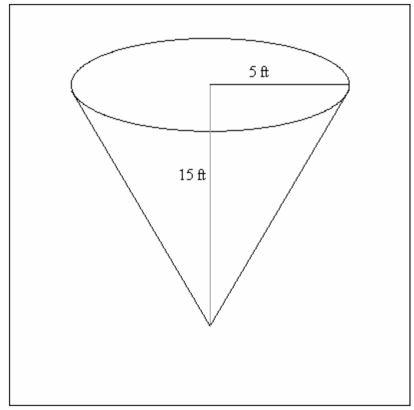
40) Since L is the line tangent to the hyperbola we know that its slope is given by $m = \frac{dy}{dx} = -\frac{6}{x^2}$ at $x = 3$, that is $m = -\frac{2}{3}$. An equation for this line is $y - 2 = -\frac{2}{3}(x - 3)$. From this equation we get that $B = 4$ and $A = 6$ which implies that the area is 12.



41) The volume of the cone is $V = \frac{1}{3}\pi r^2 h$ and the surface area is $S = \pi r^2 + \pi r\sqrt{r^2 + h^2}$. The answer is

$$\frac{S}{V} = \frac{3\left(r + \sqrt{r^2 + h^2}\right)}{rh}.$$

42) The volume of water at time t is $V(t) = 45 + 3t$. This volume occupies a cone of height $h(t)$ and radius r and therefore $45 + 3t = \frac{1}{3}\pi r^2 h$; when the tank is full $r = 5$ and $h = 15$ so $45 + 3t = \frac{1}{3}\pi 25 \times 15$ from which we get $t = \frac{125\pi - 45}{3} = 115.8996939$. The answer is 115.9



43) There are $4! = 24$ permutations that begin with 1; there are $4! = 24$ permutations that begin with a 2 and there are another 24 permutations that begin with a 3. This means that the 55th place is occupied by a permutation that begins with a 3; now, there 6 permutations that begin with 31 so we need to start with 32 (because 24+24+6 = 54). This means that the next permutation, 32145, is in the desired place.

44) Which of the following functions is equal to $f(x) = \sin^{-1}(\sin x)$ for all $-\infty < x < \infty$? The answer is none of these because $f(x) = \sin^{-1}(\sin x) = x$ only for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. Notice that this identity on this interval immediately rules out answers A), C), and D)).

45) Using the recursive formula we can write

$$\begin{aligned} f(1) &= 1 \\ f(2) &= 2f(1) + 1 = 2^2 - 1 \\ f(3) &= 2f(2) + 1 = 2^3 - 1 \\ &\vdots \\ f(n) &= 2f(n-1) + 1 = 2^n - 1 \end{aligned}$$

Then $f(1) + f(2) + \cdots\cdots + f(10) = \sum_{i=1}^{10}\left(2^i - 1\right) = 2\left(\sum_{i=1}^{10} 2^{i-1}\right) - 10 = 2\frac{1-2^9}{1-2} - 10 = 2036$

46) First recall that $\tan^{-1}(\tan x) = x$ only for $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and then notice that $\frac{3}{2}\pi < \theta < \frac{5}{2}\pi$ implies that $-\frac{\pi}{2} < \theta - 2\pi < \frac{\pi}{2}$ and because the period for the tangent function is π we know that $\tan\theta = \tan(\theta - 2\pi + 2\pi) = \tan(\theta - 2\pi)$ and therefore $\tan^{-1}(\tan\theta) = \tan^{-1}(\tan(\theta - 2\pi)) = \theta - 2\pi$.

47) If $0 < b < c$, find the value of $\cos\left(\sin^{-1}\left(\frac{b}{c}\right)\right)$. If we let $\alpha = \sin^{-1}\left(\frac{b}{c}\right)$ then we need to find $\cos\alpha$.

Now, $\alpha = \sin^{-1}\left(\frac{b}{c}\right)$ and the assumptions imply $0 \le \alpha \le \frac{\pi}{2}$ and $\sin\alpha = \frac{b}{c}$. Therefore

$$\cos\alpha = \sqrt{1 - \sin^2\alpha} = \sqrt{1 - \left(\frac{a}{b}\right)^2} = \frac{\sqrt{c^2 - b^2}}{c}.$$

48) One height of the triangle is given by $4\sin 60° = 4\frac{\sqrt{3}}{2} = 2\sqrt{3}$ and the area is $\frac{1}{2}\times 6 \times 2\sqrt{3} = 6\sqrt{3}$

49) If we subtract the first equation from the second one we get the equivalent system

$$\begin{aligned} x_1 + \qquad x_2 &= 3 \\ (a^2 - 9)x_2 &= a - 3 \end{aligned}$$

The system has no solution if it is inconsistent and that happens only when $a = -3$ because the second equation reduces to $0x_2 = -6$.

50) $$\det(B) = \begin{vmatrix} 4x & 2a & -p \\ 4y & 2b & -q \\ 4z & 2c & -r \end{vmatrix} = (4)(2)(-1)\begin{vmatrix} x & a & p \\ y & b & q \\ z & c & r \end{vmatrix} = -8\begin{vmatrix} x & y & z \\ a & b & c \\ p & q & r \end{vmatrix} = -8\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = -8\times 2 = -16$$