ROUND I - Solutions

1. In 5 years David will be twice as old as Sandra was 7 years ago. The sum of their ages is 29 years. How old is Sandra? ANSWER. 16

   Let *d* denote David's age and *s* Sandra's age. The information provided means that $d + 5 = 2(s\text{-}7)$ and $d + s = 29$. Since $d = 29 - s$ we have that $29 - s + 5 = 2s - 14$ or $3s = 48$ which means that Sandra's age is 16.

2. If you select at random an integer number from 66 to 201, both included, what is the probability that the number is divisible by 2 or 3? ANSWER. $\frac{91}{136}$

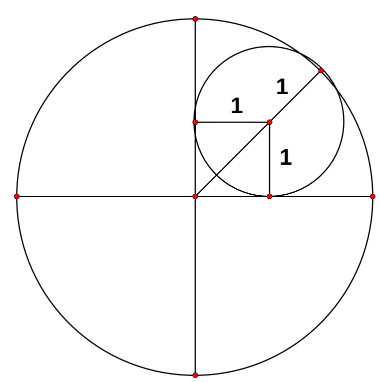
   There are 136 numbers between 66 and 201, both included. There are 100 numbers divisible by 2 between 1 and 201 and there 32 between 1 and 65; therefore there are 68 numbers divisible by 2 between 66 and 201, both included. Similarly, there 67 – 21 = 46 numbers divisible by 3 between 66 and 201, both included and there 33 – 10 = 23 numbers divisible by 6 between 66 and 201, both included. Therefore the probability that the selected number is divisible by 2 or 3 is $\frac{68}{136} + \frac{46}{136} - \frac{23}{136} = \frac{91}{136}$

3. How many permutations of the letters ABCDEF do not contain the letters A and E adjacent to each other? ANSWER. 480

   There are 6! = 720 possible permutations of the 6 given letters. The number of permutations containing the letters A and E adjacent to each other is $2 \times 5 \times 4! = 240$. Then the number of permutations of the letters ABCDEF that do not contain the letters A and E adjacent to each other is 720-240 = 480.

4. Two perpendicular lines, intersecting at the center of a circle, divide the circle into four parts. A smaller circle of radius 1 is inscribed in one of those parts as shown. What is the radius of the larger circle? ANSWER $\sqrt{2}+1$

   The figure on the right clearly implies that the radius of the larger circle is $\sqrt{2}+1$



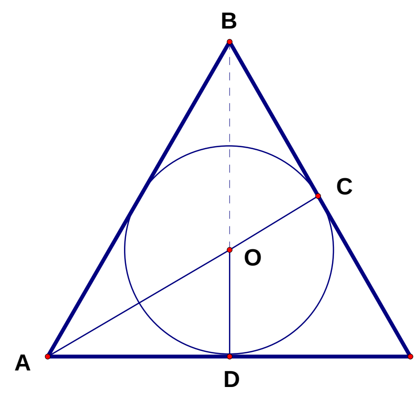
5. Let f be a function defined on the set of natural numbers given recursively by $f(1)=2$ and $f(n+1)=2f(n)-1$ for all $n \geq 1$. Find $f(11)$. ANSWER.1025

$$f(1)=2$$
$$f(2)=2f(1)-1=2\times 2-1=3=2^1+1$$
$$f(3)=2f(2)-1=2\times 3-1=5=2^2+1$$
$$\vdots$$
$$f(n)=2f(n-1)-1=2^{n-1}+1$$

This means that $f(11)=2^{10}+1=1025$

6. The length of the side of the equilateral triangle shown in the figure is 2 feet. Find the area of the inscribed circle. ANSWER. $\frac{\pi}{3}$

From the figure we conclude that $AB = 2$, $AD = 1$ and therefore $BD=\sqrt{2^2-1}=\sqrt{3}$. Since $\frac{BC}{AC}=\frac{OD}{AD}$ we have that $OD=AD\frac{BC}{AC}=\frac{1}{\sqrt{3}}$. Now, the area of the circle is $\pi(OD)^2=\frac{\pi}{3}$.



7. Find all values of x such that $0 \leq x \leq \pi$ and $\cos^4 x-\sin^4 x=0$. ANSWER. $\frac{\pi}{4},\frac{3\pi}{4}$

$\cos^4 x-\sin^4 x=\left(\cos^2 x-\sin^2 x\right)\left(\cos^2 x+\sin^2 x\right)=0$ implies $\cos^2 x-\sin^2 x=0$ or $\tan x=\pm 1$.
Since $0 \leq x \leq \pi$, the possible values of x are $\frac{\pi}{4},\frac{3\pi}{4}$.

8. Solve the equation $x+2\sqrt{x}-3=0$. ANSWER. $x=1$

Clearly the values of x and $\sqrt{x}$ must be nonnegative. Also, $x+2\sqrt{x}-3=\left(\sqrt{x}+3\right)\left(\sqrt{x}-1\right)=0$, which means $x=1$.

ROUND II- Solutions

1. Solve the equation $\sqrt{x^2-2x+1}=2x$. $\boxed{\text{ANSWER.} \quad x=\frac{1}{3}}$

   First, notice that any solution of the equation must be nonnegative. Now, if we square each side we get $x^2-2x+1=4x^2$ or $3x^2+2x-1=(3x-1)(x+1)=0$ which means that we must have $x=\frac{1}{3}$.

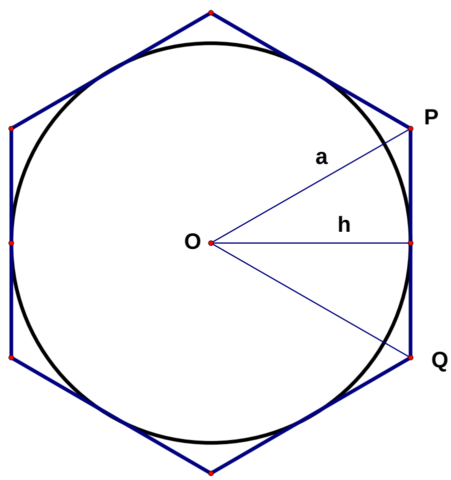
2. In order to buy a book, Amy needed 7 more cents but Bob needed 1 more cent. They decided to combine their money but even then they did not have enough money to buy the book. How much did the book cost? $\boxed{\text{ANSWER } 7.}$

   Let a,b denote the respective amount of money that Amy and Bob have and let p be the price of the book. The given information means that $\begin{matrix} a+7=p \\ b+1=p \end{matrix}$ and $a+b<p$. Adding the first two equations we get that $a+b+8=2p$; if we now use the inequality we have $2p=a+b+8<p+8$ which means $p<8$. But the first equation tells us that p must be at least 7. Thus, $p=7$.

3. If you select a number at random from the first 100 positive integers, what is the probability that the number is divisible by 3 or 4 but not both? $\boxed{\text{ANSWER. } \frac{21}{50}}$

   There are 33 numbers divisible by 3 between 1 and 100, there are 25 numbers divisible by 4 between 1 and 100, and there are 8 numbers divisible by 12 between 1 and 100. Then the desire probability is $\frac{(33-8)}{100}+\frac{(25-8)}{100}=\frac{42}{100}=\frac{21}{50}$

4. The radius of the circle inscribed in the regular hexagon shown in the figure is 2 feet. Find the area hexagon. $\boxed{\text{ANSWER. } 8\sqrt{3}}$



In an equilateral triangle of side a its height is $h=\frac{\sqrt{3}}{2}a$.

In our case $2=\frac{\sqrt{3}}{2}a$ or $\frac{4}{\sqrt{3}}=a$. This means that the area of the hexagon is $6\times\frac{1}{2}\times 2\times\frac{4}{\sqrt{3}}=8\sqrt{3}$.

5. Let f be a function defined for all real numbers with the properties that $f(0)=1$, $f(1)=2$ and $f(x+y)=f(x)f(y)$ for all x, y. Find $f(-2)$. ANSWER. $\frac{1}{4}$

First, $1=f(2+(-2))=f(2)f(-2)$ which implies that $f(-2)=\frac{1}{f(2)}=\frac{1}{f(1+1)}=\frac{1}{f(1)f(1)}=\frac{1}{4}$

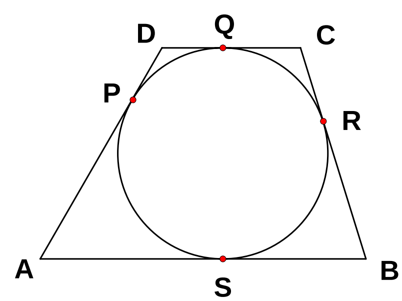
6. A bag contains a penny, a nickel, a dime, a quarter, a half-dollar coin and a dollar coin. Three coins are selected (without replacement) at random and their values added. How many possible sums are there?

ANSWER. 20

There are 6 coins in the bag. The order in which the coins are selected is not important because we are interested in their sum. The answer is given by $\binom{6}{3}=\frac{6!}{3!(6-3)!}=20$.

7. A circle is inscribed in quadrilateral $ABCD$ as shown, with $AB=17$ and $DC=12$. What is the perimeter of the quadrilateral?

ANSWER. 58



From the picture we have $DP=DQ$, $CQ=CR$, $AP=AS$, and $BR=BS$. Since the perimeter is given by $AP+PD+DQ+QC+CR+RB+BS+SA=$
$2(BS+SA)+2(DQ+QC)=34+24=58$

8. If $x<-2$, find an equivalent expression for $\left|1-\left|1+x\right|\right|$ that does not utilize any absolute value signs.
ANSWER. $-x-2$

Since $x+1<-1$, we have $\left|1-(-(1+x))\right|=\left|2+x\right|=-2-x$, where the last equality comes from the fact that $x<-2$.