Thirtieth Annual Columbus State Invitational Mathematics Tournament

Sponsored by
The Columbus State University
Department of Mathematics
March 6^{th}, 2004;

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Solutions

1. If we let x be the other number, then $\dfrac{x+(2a+3)}{2} = 5a$, which yields $x = 8a - 3$ so the answer is (B).

2. The probability to throw a head (or tails) is $\dfrac{1}{2}$ when the coin is flipped once. The probability that no head will be thrown is $\dfrac{1}{2^3}$ when a fair coin is flipped three times. Therefore the probability that at least one head will be thrown is $1 - \dfrac{1}{2^3} = 0.875$. So, the correct answer is (A).

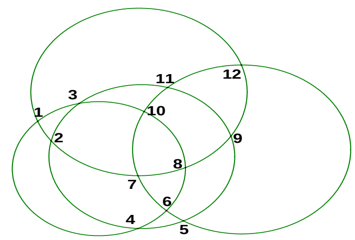
3. Let x be the length of the edge of the original cube. The surface area of the new cube is $6(x+2)^2$ square inches. Hence $6(x+2)^2 = 486$ and $x = 7$. Answer: (C).

4. $\left(\sqrt{3}^{\sqrt{3}}\right)^{\sqrt{3}} = (\sqrt{3})^3 = (\sqrt{3})^2\sqrt{3} = 3\sqrt{3}$. Hence the answer is (D).

5. The pool is leaking at a rate of 0.3 cubic feet per minute which is equivalent to 18 cubic feet per hour. A parallelipiped which is 12 feet wide, 18 feet long and 1 food deep has volume $12 \times 18 \times 1$ ft^3. Using volume=time×rate, we see that (B) is the correct answer.

6. $A = \dfrac{1}{y} + \dfrac{1}{2y^2} + \dfrac{1}{2yx} - \dfrac{1}{2y^2} = \dfrac{1}{y}\left(1 + \dfrac{1}{2x}\right)$ and $B = \dfrac{1}{2xy^2}\left(1 + \dfrac{1}{2x}\right)$. So $A^2/B = \dfrac{\frac{1}{y^2}\left(1+\frac{1}{2x}\right)^2}{\left(1+\frac{1}{2x}\right)} 2xy^2 = 2x + 1$. Answer: (D).

7. Two different circles may intersect at not more than two points. Given four circles there are $\binom{4}{2} = 6$ different pairs that can be formed. Each of these may give rise to two different intersection points. This gives a maximum of $6 \times 2 = 12$ intersections. To see that this maximumu is really possible see the acompanying figure. Answer: A.

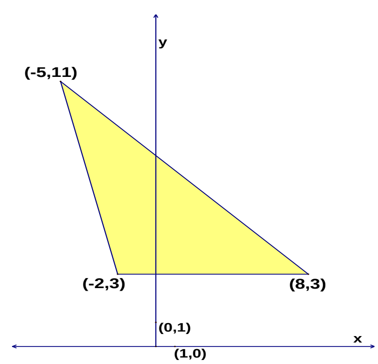


8. Grouping the terms at the denominator and at the numerator, we have

$$\frac{(3-6)+(9-12)+\cdots+(2001-2004)}{(7-14)+(21-28)+\cdots+(4669-4676)} = \frac{\overbrace{(-3)+(-3)+\cdots+(-3)}^{334\ times}}{\underbrace{(-7)+(-7)+\cdots+(-7)}_{334\ times}} = \frac{(-3)\times 334}{(-7)\times 334} = \frac{3}{7}.$$

Answer: (D).

9. Since there are two vertices of this triangle with the same y-coordinate we can use the side determined by them as base to obtain the area $(8-(-2)) \times (11-3)/2 = 40$ so the correct answer is B.



10. Note that $0 < \frac{1}{2} + \frac{1}{3} + \frac{1}{n} \leq \frac{5}{6} + 1 < 2$. Since the sum of these three fractions is an integer, it must be the integer 1. The equation $\frac{1}{2} + \frac{1}{3} + \frac{1}{n} = 1$ implies that $n = 6$. In this case we see that (E) is the correct answer.

11. Let x be the smallest of the 9 consecutive positive integers. Then the sum $x + (x+1) + \cdots + (x+8) = 9x + 36$. Note that $9x + 36$ is divisible by 9 and out of the given answers only 225 satisfies this condition. If $x = 21$ then $9x + 36 = 225$. So the correct choice here is (A).

12. Using the formula $1 + x + x^2 + \cdots + x^n = \dfrac{x^{n+1} - 1}{x - 1}$ with $x = 2$ and $n = 100$, we have the answer $2^{101} - 1$. So the answer is (D).

13. We observe that the k^{th} term in the sum is $a_k = 3k + 10$ which give the first term when $k = 1$ and the last term when $k = 30$. We have then 30 terms in the sum. Then the sum is $\dfrac{30(13 + 100)}{2} = 1695$. Answer:(C).

14. The sum

$$\sum_{k=1}^{2004} \left(\frac{1}{5} k + 1 \right) = \frac{1}{5} \sum_{k=1}^{2004} k + \sum_{k=1}^{2004} 1 = \frac{1}{5} \frac{(2004)(1 + 2004)}{2} + 2004 = 403806.$$

Hence the correct answer is (A).

15. Note that $g(0.4) = g\left(\dfrac{2}{5}\right) = \dfrac{9}{70} < \dfrac{1}{7}$ and $g(g(0.4)) = \dfrac{33}{35} > \dfrac{1}{7}$. Hence $g(g(g(0.4))) = \frac{1}{2}(\frac{33}{35} - \frac{1}{7}) = \frac{2}{5} = 0.4$. Answer: (E).

16. Since the expression is defined for any numbers a and b, choose $a = 0$ and $b = 0$. We see that $E = 1$. So the answer must be (A). In fact the following identity takes place

$$(a + b + c)^3 - a^3 - b^3 - c^3 = 3(a + b)(a + c)(b + c).$$

17. Note that $4 \times \underbrace{166\ldots66}_{n\ digits} = \underbrace{666\ldots64}_{n\ digits}$ which means $F = \frac{1}{4}$ for any number of digits $n \geq 2$. So the number n is undetermined. Answer: (B).

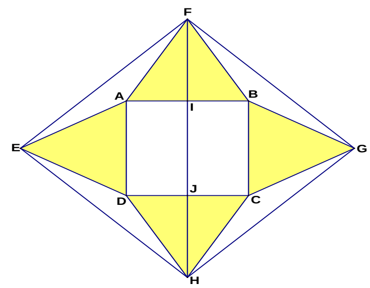
18. Note that $0 < \cos t = \dfrac{x}{1 - x} < 1$ implies $0 < x < 1/2$. Hence $\sin t = \sqrt{1 - (\cos t)^2} = \sqrt{1 - [x/(1 - x)]^2} = \frac{\sqrt{1-2x}}{1-x}$. Thus $\tan t = \dfrac{\sin t}{\cos t} = \dfrac{\sqrt{1 - 2x}}{x}$. Answer: (B).

19. Using the addition formula $\cos(A + B) = \cos A \cos B - \sin A \sin B$ with $A = x$ and $B = x - y$, we obtain $E = \cos(2x - y)$. Answer:(C).

20. Method I: We have $(3x - 1)(ax^2 + bx + c) = 3ax^3 + (3b - a)x^2 + (3c - b)x - c$. Then $12x^3 - 40x^2 + 27x - 5 = 3ax^3 + (3b - a)x^2 + (3c - b)x - c$ implies that $3a = 12, 3b - a = -40$, $3c - b = 27$. So $a = 4, b = -12$ and $c = 5$, which gives (E) as the correct choice.

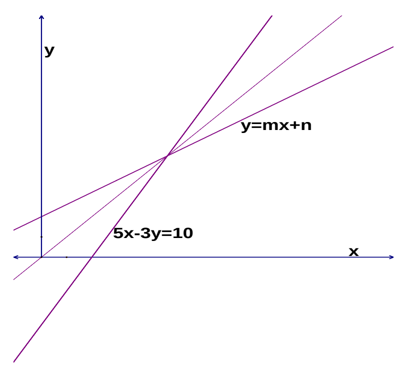
Method II: Calculate $P(1) = 12 - 40 + 27 - 5 = (3 \times 1 - 1)(a + b + c)$. Hence $-6 = 2(a + b + c)$ which implies $a + b + c = -3$.

21. Method I. Let I and J be the intersections of the diagonal $\overline{FH}$ with $\overline{AB}$ and $\overline{CD}$. These two points are the midpoints of $\overline{AB}$ and $\overline{CD}$. Hence $FH = FI + IJ + JH = IJ + 2 \times FI = 2 + 2 \times \frac{\sqrt{3}}{2} \times 2$ because the height of an equilateral triangle with base a is $\frac{\sqrt{3}}{2}a$. The diagonal of a square with side lengths s is $s\sqrt{2}$. Thus we can then find $EF = (2 + 2\sqrt{3})/\sqrt{2} = \sqrt{2} + \sqrt{6}$ so the answer is B.



Method II. Since $m\angle FAE = 360° - (2 \times 60° + 90°) = 150°$, we get that $m\angle EFA = 15°$. So, $EF = 2 \times 2\cos(15°)$ or $EF = 4\sqrt{\frac{1+\cos 30°}{2}} = \sqrt{8 + 4\sqrt{3}} = \sqrt{2} + \sqrt{6}$.

22. The function whose graph is symmetric with respect to the line $y = x$ must be the inverse function. So interchanging x and y and then solving for y we obtain $y = \frac{3x}{5} + 2$. Therefore $n = 2$ and the answer is D.



23. $a^2 = b^2$ if and only if $a = b$ or $a = -b$. The equation $x^2 + 4x - 2 = 5x^2 - 1$ has a repeated solution equal to $\frac{1}{2}$. The equation $x^2 + 4x - 2 = -(5x^2 - 1)$ has two distinct real solutions $x_{1,2} = -\frac{1}{3} \pm \frac{\sqrt{22}}{6}$. Therefore there are three different real solutions of our original equation, which says that the correct answer is (B).

24. Let x be the new price of apples in cents per dozen. The number of apples purchased with 50 cents at the price x is the same as the number of apples purchased with 50 cents at the price $x + 10$ in the previous week plus 5 more apples. That is, $12 \times \frac{50}{x} = 5 + 12 \times \frac{50}{x + 10}$, which gives $x = 30$. Answer: (E).

25. Let x_1, x_2, x_3, x_4 and x_5 be the five solutions of a polynomial equation of the form:

$$x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0.$$

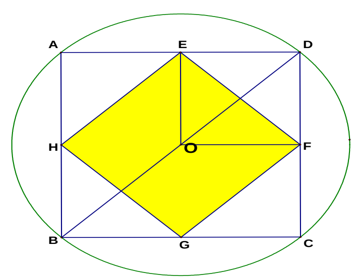
Then we have

$$x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = (x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5).$$

After expanding the right hand side of the above equality, we observe that $x_1 + x_2 + x_3 + x_4 + x_5 = -a_4$. The terms involving of x^4 in $(x-1)^5 + (x-2)^4 + (x-3)^3 + (x-4)^2 + (x-5)$ are $-5x^4 + x^4 = -4x^4$. So the sum of the five solutions is 4. Therefore the answer is (D).

26. Observe that for every k between 2 and 2004 we have $1 - \dfrac{1}{k^2} = \dfrac{k^2-1}{k^2} = \dfrac{(k-1)}{k}\dfrac{(k+1)}{k}$. This means that the given product is the same as multiplying $\dfrac{1}{2}\dfrac{2}{3}\cdots\dfrac{2003}{2004}$ with $\dfrac{3}{2}\dfrac{4}{3}\cdots\dfrac{2005}{2004}$. After simplifications the first product is equal to $\frac{1}{2004}$ and the second is equal to $\frac{2005}{2}$. Hence the answer is $\frac{2005}{4008}$ which is given in choice (B).

27. Construct the diagonal $\overline{BD}$ and and let O be the center of the circle. Then $EF = OD = 6\ cm$ and so the area of $EFGH$ is $6 \times 6 = 36\ cm^2$. Answer (E).

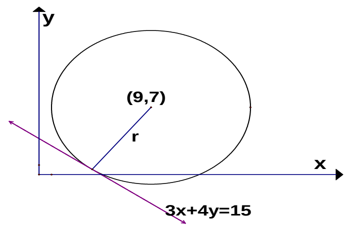


28. Let the three solutions be $x_1 = u - r$, $x_2 = u$ and $x_3 = u + r$. Since their sum is suppose to be 12 we obtain $u = 4$. From the fact that $(x - x_1)(x - x_2)(x - x_3) = x^3 - 12x^2 + 44x + a$ we obtain that $x_1x_2 + x_2x_3 + x_3x_1 = (u-r)u + u(u+r) + (u+r)(u-r) = 44$. Solving for r we get $r = \pm 2$. Both values of r give the same set of solutions: 2, 4 and 6. Hence $a = -x_1x_2x_3 = -2 \times 4 \times 6 = -48$. Answer: (B).

29. Method I: One can do the long division and obtain $x^4 - 5x^3 + 14x^2 - 15x + 10 = (x^2 - 3x + 7)(x^2 - 2x + 1) + 2x + 3$. Answer: (D).

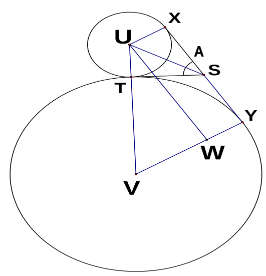
Method II: Let $P(x) = x^4 - 5x^3 + 14x^2 - 15x + 10$ and suppose $P(x) = Q(x)(x-1)^2 + ax + b$. Letting $x = 1$ we obtain $P(1) = a + b$. So $a + b = 5$. Notice that $P'(1) = a$. Therefore $a = 4 - 5 \times 3 + 14 \times 2 - 15 = 2$ and then $b = 3$.

30. The equation of the line perpendicular to $3x + 4y = 15$ is $y - 7 = \frac{4}{3}(x - 9)$ or $4x - 3y = 15$. These two lines intersect at $(21/5, 3/5)$ and then the radius is $\sqrt{(9 - 21/5)^2 + (7 - 3/5)^2} = 8$. Answer: (C).

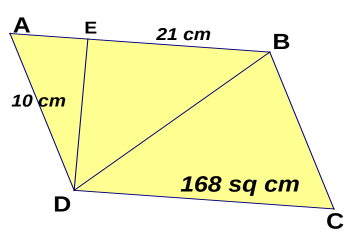


31. We notice that choice (A) means that $(x^y)^z = x^{y^z}$ or $x^{yz} = x^{y^z}$ which is not true for every x, y, z. The second choice means $(xy)^z = x^z y^z$ which is a standard formula. One can check that for each of the other choices there are counterexamples to the proposed "identity". Hence the answer should be (B).

32. Let X, Y and T be the points of tangency as in the accompanying figure. Denote by U and V the centers of the two circles and construct $\overline{UW}$ parallel to $\overline{XY}$. Since $UV = 9 + 3 = 12$ and $VW = 9 - 3 = 6$ the measure of $\angle UVW = \pi/3$. But $\angle A = \angle UVW$ because the quadrilateral VTSY is incribable ($m\angle VTS = m\angle VYS = \pi/2$). Answer: D.



33. Let E on $\overline{AB}$ such that $\overline{DE}$ is perpendicular to $\overline{AB}$. (It may happen that the perpendicular form D on $\overline{AB}$ does not intersect $\overline{AB}$ but in this case we consider the perpendicular at A on $\overline{AB}$). Using the formula for the area of a parallelogram we get $DE = 168/21 = 8\ cm$. Then $AE = 6\ cm$, $EB = 21 - 6 = 15\ cm$ and so $BD = \sqrt{8^2 + 15^2} = 17\ cm$. Answer: A.



34. Using the change of base formula we get $\log_k x = \frac{\log_7 x}{\log_7 k}$. This implies that

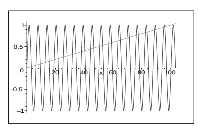
$$\log_7 x = (\log_k x)(\log_7 k) = 3.$$

Hence $x = 7^3 = 343$. Answer: (B).

35. Let us observe that 2^x is around 2004 if $x = \frac{\ln 2004}{\ln 2} \approx 10.96$. In fact $2^{11} - 4 \times 11 = 2004$. So, one solution is $x_1 = 11$. We see that $-4x = 2004$ implies $x = -501$ and 2^{-501} is almost 0 ($\approx .15 \times 10^{-150}$) which suggests that $x_2 \approx -501$. Hence the sum of the two solutions must be very close to $-501 + 11 = -490$. Answer: (D).

36. The side lengths of this triangle must be $3a$, $4a$, $5a$, and so its area is $6a^2 = 726$. This equation gives $a = 11$ *in* and then the hypotenuse is 55 *in*. Answer: (D).

37. Let us observe that $\sin x = 1$ for $x = \pi/2 + 2k\pi$ where k is an integer. So the intersection of $\sin x$ and $x/2004$ is going to be around these values of x for x large (see figure). In fact we are going to have intersections around $x_k = \pi/2 + 2k\pi$ as long as $\frac{x_k}{2004} < 1$. The greatest k with this property is 318. Using a calculator one can check that $(2\pi \times 319 + \pi/4)/2004 > 1$. This means that we cannot have any other solution of our equation larger than $2\pi \times 319$. Then for each k from 0 to 318 there are two solutions around x_k. Excluding 0 we get a total of $2 \times 319 - 1 = 637$. Answer: (A).



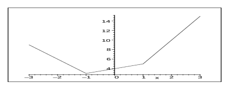
38. The equation can be written as $\dfrac{x-3}{2} + \dfrac{x-2}{3} = \dfrac{2}{x-3} + \dfrac{3}{x-2}$ or $\dfrac{5x-13}{6} = \dfrac{5x-13}{(x-3)(x-2)}$. So, another solution of this equation is $x_2 = \frac{13}{5}$ and solving $(x-3)(x-2) = 6$ one gets $x_1 = 0$ and $x_3 = 5$. Thus $x_2 x_3 = 13$. Answer: (E).

39. If $N = abcdef$ with all digits different the number of possibilities would have been $6!$. But if $a = b$ the number of possibilities reduces to just a half because, for instance, $fcadbe$ and $fcbdae$ would count only as one permutation instead of two. For the same reasoning since $c = d$ and $e = f$ the number of all diffrent permutation derived from N is $6!/2^3 = 90$. Answer: (A).

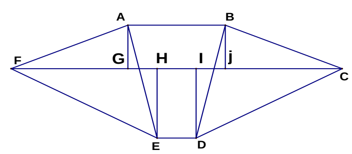
40. Let us observe that the inequality makes sense only if $1 - x^3 \geq 0$ which means $x \leq 1$ and $1 - \sqrt{1 - x^3} \geq 0$ which turns out to $x \geq 0$. So assuming that $0 \leq x \leq 1$ we can safely

square both sides of the given inequality and obtain $1 - x^2 \geq \sqrt{1 - x^3}$. Squaring again we get equivalently $x^2(x + 2)(x - 1) \geq 0$. Since $0 \leq x \leq 1$ this is possible if and only if $x = 0$ or 1. Answer: (B).

41. The graph of $f(x) = x + 2|x + 1| + 2|x - 1|$ is shown in the accompanying figure. The problem is just another way of asking for the minimum of this function which is $f(-1) = 3$. Answer: (A).

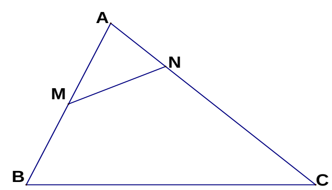


42. Let G, H, I, and J be the projections of A, E, D and B on $\overline{FC}$. Then $FG = JC = (FC - GJ)/2 = 12\ in.$ From the right triangle BJC we obtain $BJ = \sqrt{13^2 - 12^2} = 5$. Similarly $FH = IC = (FC - HI)/2 = 15$ and $ID = \sqrt{17^2 - 15^2} = 8$. Then the area of $ABDE$ is $(10 + 4) \times (5 + 8)/2 = 91\ in^2$. Answer: E.



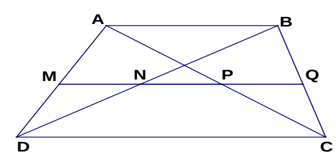
43. Let $g(x) = (x^2 + x - 1)^{x-3}$ defined on integers. We observe that $g(-2)$, $g(-1)$, $g(1)$ and $g(3)$ are all equal to 1. On the other hand we get $g(0) = -1$ and $g(2) = 1/5$. If $x \geq 4$ we can be sure that $g(x) > 1$ because $x^2 + x - 1 > 1$ and $x - 3 > 0$. Similarly, if $x \leq -3$ then $g(x) < 1$ because $x^2 + x - 1 > 1$ and $x - 3 < 0$. Hence the answer is (E).

44. Triangles AMN and ACB are similar since $m\angle MAN = m\angle CAB$ and $m\angle AMN = m\angle ACB$. Hence $\frac{AM}{AC} = \frac{MN}{BC}$ which gives $MN = (18 \times 7)/21 = 6\ in$. Answer: A.



45. Applying logarithms both sides of $a^{2b} = b^{3a}$ we obtain $2b \ln a = 3a \ln b$. Substituting $b = 5a$ in this equation we get $10a \ln a = 3a(\ln 5 + \ln a)$. Simplifying by $a(a > 0)$ and then solving for a one arrives at $a = 5^{3/7}$. Answer: (C).

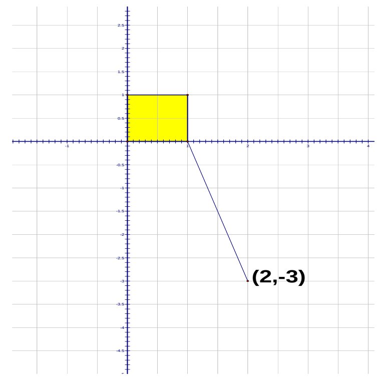
46. Let $MN = x$. Triangles DMN and DAB are similar. So $\frac{x}{2} = \frac{ND}{BD}$. From the similarity of triangles BNQ and BDC we get $\frac{2x}{6} = \frac{BN}{BD}$. Adding these two equalities together we obtain $x(1/2 + 1/3) = 1$ which gives $x = 6/5 = 1.2\ cm$. Answer: C.



47. Method I: Let $f(t) = 3t^2 - t - 1$. This quadratic function has two different zeroes in the interval $(-1, 1)$ because $f(-1) = 3 > 0$, $f(0) = -1 < 0$ and $f(1) = 1 > 0$. These two different zeroes are not opposite to one another and so the quadratic equation in $\cos(2x) = 2\cos(x)^2 - 1$ has two different solutions. This means that the problem is determined uniquely because the free coefficient is -1. To determine the equation in $\cos 2x$ one can proceed as follows. From the given equation we have $3(1 + \cos 2x) - 2\cos x - 2 = 0$ or $2\cos x = 1 + 3\cos 2x$. Squaring both sides one gets $4(\cos x)^2 (= 2 + 2\cos 2x) = 9(\cos 2x)^2 + 6\cos 2x + 1$ or $9(\cos 2x)^2 + 4\cos 2x - 1 = 0$. Since the solution should be unique $a = 9$ and $b = 4$. Answer: D.

Method II: One can solve the quadratic $f(t) = 0$ and find the solutions $t_{1,2} = \frac{1 \pm \sqrt{13}}{6}$ then substitute in the double angle formula to obtain the solutions in $\cos 2x$: $\frac{-2 \pm \sqrt{13}}{9}$. Since their product is $-1/9$, a must be 9 and because their sum is $\frac{-4}{9}$, b must be 4. Hence answer is (D).

48. Write f as $f(x,y) = (x-2)^2 + (y+3)^2 - 13$. Using the distance formula between two points $d(x,y) := d((x,y),(2,-3)) = \sqrt{(x-2)^2+(y+3)^2}$ we see that $f(x,y) = d(x,y)^2 - 13$. So, to minimize f over the unit square is the same as minimizing the distance from a point in this square to the point of coordinates $(2,-3)$. The closest point in the square to $(2,-3)$ is then $(1,0)$. So $f(1,0) = -3$ must be the minimum. Answer: C.



49. The slope of the tangent line is $\frac{dy}{dx}|_{x=1} = 2x_{|x=1} = 2$ so the equation of this tangent line is $y - 1 = 2(x-1)$ which means $y = 2x - 1$. Therefore the y-intercept is -1. Answer: E.

50. $\lim_{x\to 0} \frac{3^x - 1}{2^x - 1} = \lim_{x\to 0} \frac{\frac{3^x-1}{x}}{\frac{2^x-1}{x}} = \frac{\frac{d\,3^x}{dx}|_{x=0}}{\frac{d\,2^x}{dx}|_{x=0}} = \frac{3^0 \ln 3}{2^0 \ln 2} = \frac{\ln 3}{\ln 2} = \log_2 3$. Answer: A.