Solutions

1. If we let \( x \) be the other number, then \( \frac{x + (2a + 3)}{2} = 5a \), which yields \( x = 8a - 3 \) so the answer is (B).

2. The probability to throw a head (or tails) is \( \frac{1}{2} \) when the coin is flipped once. The probability that no head will be thrown is \( \frac{1}{2^3} \) when a fair coin is flipped three times. Therefore the probability that at least one head will be thrown is \( 1 - \frac{1}{2^3} = 0.875 \). So, the correct answer is (A).

3. Let \( x \) be the length of the edge of the original cube. The surface area of the new cube is \( 6(x + 2)^{2} \) square inches. Hence \( 6(x + 2)^{2} = 486 \) and \( x = 7 \). Answer: (C).

4. \( \left( \sqrt[3]{3} \right)^{\sqrt[3]{3}} = \left( \sqrt[3]{3} \right)^{3} = \sqrt[3]{3} \cdot \sqrt[3]{3} \cdot \sqrt[3]{3} = 3\sqrt[3]{3} \). Hence the answer is (D).

5. The pool is leaking at a rate of 0.3 cubic feet per minute which is equivalent to 18 cubic feet per hour. A parallelepiped which is 12 feet wide, 18 feet long and 1 foot deep has volume \( 12 \times 18 \times 1 \) ft\(^3\). Using volume=rate \times \text{time}, we see that (B) is the correct answer.

6. \( A = \frac{1}{y} + \frac{1}{2y^{2}} + \frac{1}{2y^{2}x} - \frac{1}{2y^{2}} = \frac{1}{y} \left( 1 + \frac{1}{2x} \right) \) and \( B = \frac{1}{2xy^{2}} \left( 1 + \frac{1}{2x} \right) \). So \( A^{2}/B = \frac{\frac{1}{y} \left( 1 + \frac{1}{2x} \right)^{2}}{2xy^{2}} = \frac{1}{2x + 1} \). Answer: (D).
7. Two different circles may intersect at not more than two points. Given four circles there are \( \binom{4}{2} = 6 \) different pairs that can be formed. Each of these may give rise to two different intersection points. This gives a maximum of \( 6 \times 2 = 12 \) intersections. To see that this maximum is really possible see the accompanying figure. Answer: A.

8. Grouping the terms at the denominator and at the numerator, we have

\[
\frac{(3 - 6) + (9 - 12) + \cdots + (2001 - 2004)}{(7 - 14) + (21 - 28) + \cdots + (4669 - 4676)} = \frac{(-3) + (-3) + \cdots + (-3)}{(-7) + (-7) + \cdots + (-7)} = \frac{(-3) \times 334}{(-7) \times 334} = \frac{3}{7}.
\]

Answer: (D).

9. Since there are two vertices of this triangle with the same y-coordinate we can use the side determined by them as base to obtain the area \((8 - (-2)) \times (11 - 3)/2 = 40\) so the correct answer is B.

10. Note that \(0 < \frac{1}{2} + \frac{1}{3} + \frac{1}{n} \leq \frac{5}{6} + 1 < 2\). Since the sum of these three fractions is an integer, it must be the integer 1. The equation \(\frac{1}{2} + \frac{1}{3} + \frac{1}{n} = 1\) implies that \(n = 6\). In this case we see that (E) is the correct answer.

11. Let \(x\) be the smallest of the 9 consecutive positive integers. Then the sum \(x + (x + 1) + \cdots + (x + 8) = 9x + 36\). Note that \(9x + 36\) is divisible by 9 and out of the given answers only 225 satisfies this condition. If \(x = 21\) then \(9x + 36 = 225\). So the correct choice here is (A).
12. Using the formula \(1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}\) with \(x = 2\) and \(n = 100\), we have the answer \(2^{101} - 1\). So the answer is (D).

13. We observe that the \(k^{th}\) term in the sum is \(a_k = 3k + 10\) which give the first term when \(k = 1\) and the last term when \(k = 30\). We have then 30 terms in the sum. Then the sum is \(\frac{30(33 + 10)}{2} = 1695\). Answer: (C).

14. The sum
\[
\sum_{k=1}^{2004} \left( \frac{1}{5} k + 1 \right) = \frac{1}{5} \sum_{k=1}^{2004} k + \sum_{k=1}^{2004} 1 = \frac{1}{5} \frac{(2004)(1 + 2004)}{2} + 2004 = 403806.
\]
Hence the correct answer is (A).

15. Note that \(g(0.4) = g\left(\frac{2}{5}\right) = \frac{9}{70} < \frac{1}{7}\) and \(g(g(0.4)) = \frac{33}{35} > \frac{1}{7}\). Hence \(g(g(g(0.4)))) = \frac{1}{2} \left(\frac{33}{35} - \frac{1}{7}\right) = \frac{2}{5} = 0.4\). Answer: (E).

16. Since the expression is defined for any numbers \(a\) and \(b\), choose \(a = 0\) and \(b = 0\). We see that \(E = 1\). So the answer must be (A). In fact the following identity takes place
\[
(a + b + c)^3 - a^3 - b^3 - c^3 = 3(a + b)(a + c)(b + c).
\]

17. Note that \(4 \times 166\ldots 66 = 666\ldots 64\) which means \(F = \frac{1}{4}\) for any number of digits \(n \geq 2\). So the number \(n\) is undetermined. Answer: (B).

18. Note that \(0 < \cos t = \frac{x}{1 - x} < 1\) implies \(0 < x < 1/2\). Hence \(\sin t = \sqrt{1 - \cos^2 t} = \sqrt{1 - \left|\frac{x}{1-x}\right|^2} = \frac{\sqrt{1-2x}}{1-x}\). Thus \(\tan t = \frac{\sin t}{\cos t} = \frac{\sqrt{1-2x}}{x}\). Answer: (B).

19. Using the addition formula \(\cos(A + B) = \cos A \cos B - \sin A \sin B\) with \(A = x\) and \(B = x - y\), we obtain \(E = \cos(2x - y)\). Answer: (C).

20. Method I: We have \((3x - 1)(ax^2 + bx + c) = 3ax^3 + (3b - a)x^2 + (3c - b)x - c\). Then \(12x^3 - 40x^2 + 27x - 5 = 3ax^3 + (3b - a)x^2 + (3c - b)x - c\) implies that \(3a = 12, 3b - a = -40, 3c - b = 27\). So \(a = 4, b = -12\) and \(c = 5\), which gives (E) as the correct choice.
Method II: Calculate $P(1) = 12 - 40 + 27 - 5 = (3 \times 1 - 1)(a + b + c)$. Hence $-6 = 2(a + b + c)$ which implies $a + b + c = -3$.

21. Method I. Let $I$ and $J$ be the intersections of the diagonal $FH$ with $AB$ and $CD$. These two points are the midpoints of $AB$ and $CD$. Hence $FH = FI + IJ + JH = IJ + 2 \times FI = 2 + 2 \times \frac{\sqrt{3}}{2} \times 2$ because the height of an equilateral triangle with base $a$ is $\frac{\sqrt{3}}{2}a$. The diagonal of a square with side lengths $s$ is $s\sqrt{2}$. Thus we can then find $EF = \left(2 + 2\sqrt{3}\right) / \sqrt{2} = \sqrt{2} + \sqrt{6}$ so the answer is B.

Method II. Since $m\angle FAE = 360^\circ - (2 \times 60^\circ + 90^\circ) = 150^\circ$, we get that $m\angle EFA = 15^\circ$. So, $EF = 2 \times 2 \cos(15^\circ)$ or $EF = 4 \sqrt{1 + \cos{30^\circ}} = \sqrt{8 + 4 \sqrt{3}} = \sqrt{2} + \sqrt{6}$.

22. The function whose graph is symmetric with respect to the line $y = x$ must be the inverse function. So interchanging $x$ and $y$ and then solving for $y$ we obtain $y = \frac{3x}{5} + 2$. Therefore $n = 2$ and the answer is D.

23. $a^2 = b^2$ if and only if $a = b$ or $a = -b$. The equation $x^2 + 4x - 2 = 5x^2 - 1$ has a repeated solution equal to $\frac{1}{2}$. The equation $x^2 + 4x - 2 = -(5x^2 - 1)$ has two distinct real solutions $x_{1,2} = \frac{-1}{3} \pm \frac{\sqrt{22}}{6}$. Therefore there are three different real solutions of our original equation, which says that the correct answer is (B).

24. Let $x$ be the new price of apples in cents per dozen. The number of apples purchased with 50 cents at the price $x$ is the same as the number of apples purchased with 50 cents at the price $x + 10$ in the previous week plus 5 more apples. That is, $12 \times \frac{50}{x} = 5 + 12 \times \frac{50}{x + 10}$, which gives $x = 30$. Answer: (E).
25. Let \( x_1, x_2, x_3, x_4 \) and \( x_5 \) be the five solutions of a polynomial equation of the form:
\[
x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0.
\]

Then we have
\[
x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = (x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5).
\]

After expanding the right hand side of the above equality, we observe that \( x_1 + x_2 + x_3 + x_4 + x_5 = -a_4 \). The terms involving of \( x^4 \) in \((x - 1)^5 + (x - 2)^4 + (x - 3)^3 + (x - 4)^2 + (x - 5)\) are \(-5x^4 + x^4 = -4x^4\). So the sum of the five solutions is 4. Therefore the answer is (D).

26. Observe that for every \( k \) between 2 and 2004 we have
\[
1 - \frac{1}{k^2} = \frac{k^2 - 1}{k^2} = \frac{(k - 1)(k + 1)}{k^2}.
\]

This means that the given product is the same as multiplying \( \frac{1 \cdot 2 \cdot 3 \cdots 2003}{2004} \) with \( \frac{3 \cdot 4 \cdots 2005}{2004} \). After simplifications the first product is equal to \( \frac{1}{2004} \) and the second is equal to \( \frac{2005}{2} \). Hence the answer is \( \frac{2005}{4008} \) which is given in choice (B).

27. Construct the diagonal \( BD \) and and let \( O \) be the center of the circle. Then \( EF = OD = 6 \text{ cm} \) and so the area of \( EFGH \) is \( 6 \times 6 = 36 \text{ cm}^2 \). Answer (E).

28. Let the three solutions be \( x_1 = u - r, \ x_2 = u \) and \( x_3 = u + r \). Since their sum is suppose to be 12 we obtain \( u = 4 \). From the fact that \((x - x_1)(x - x_2)(x - x_3) = x^3 - 12x^2 + 44x + a \) we obtain that \( x_1x_2 + x_2x_3 + x_3x_1 = (u - r)u + u(u + r) + (u + r)(u - r) = 44 \). Solving for \( r \) we get \( r = \pm 2 \). Both values of \( r \) give the same set of solutions: 2, 4 and 6. Hence \( a = -x_1x_2x_3 = -2 \times 4 \times 6 = -48 \). Answer: (B).

29. Method I: One can do the long division and obtain \( x^4 - 5x^3 + 14x^2 - 15x + 10 = (x^2 - 3x + 7)(x^2 - 2x + 1) + 2x + 3 \). Answer: (D).

Method II: Let \( P(x) = x^4 - 5x^3 + 14x^2 - 15x + 10 \) and suppose \( P(x) = Q(x)(x-1)^2 + ax+b \). Letting \( x = 1 \) we obtain \( P(1) = a + b \). So \( a + b = 5 \). Notice that \( P'(1) = a \). Therefore \( a = 4 - 5 \times 3 + 14 \times 2 - 15 = 2 \) and then \( b = 3 \).
30. The equation of the line perpendicular to 
\[3x + 4y = 15\] is 
\[y - 7 = \frac{4}{3}(x - 9)\] or 
\[4x - 3y = 15.\] These two lines intersect at \((21/5, 3/5)\) and then the radius is 
\[\sqrt{(9 - 21/5)^2 + (7 - 3/5)^2} = 8.\] Answer: (C).

31. We notice that choice (A) means that \((xy)^z = x^y z\) or \(x^y z = x^y z\) which is not true for every \(x, y, z\). The second choice means \((xy)^z = x^z y^z\) which is a standard formula. One can check that for each of the other choices there are counterexamples to the proposed “identity”. Hence the answer should be (B).

32. Let \(X, Y\) and \(T\) be the points of tangency as in the accompanying figure. Denote by \(U\) and \(V\) the centers of the two circles and construct \(UW\) parallel to \(XY\). Since \(UV = 9 + 3 = 12\) and \(VW = 9 - 3 = 6\) the measure of \(\angle UVW = \pi/3\). But \(\angle A = \angle UVW\) because the quadrilateral \(VTSY\) is incircleable \((m\angle VT S = m\angle VY S = \pi/2)\). Answer: D.

33. Let \(E\) on \(AB\) such that \(DE\) is perpendicular to \(AB\). (It may happen that the perpendicular form \(D\) on \(AB\) does not intersect \(AB\) but in this case we consider the perpendicular at \(A\) on \(AB\)). Using the formula for the area of a parallelogram we get \(DE = 168/21 = 8\) cm. Then \(AE = 6\) cm, \(EB = 21 - 6 = 15\) cm and so \(BD = \sqrt{8^2 + 15^2} = 17\) cm. Answer: A.
34. Using the change of base formula we get \( \log_k x = \frac{\log_7 x}{\log_7 k} \). This implies that

\[
\log_7 x = (\log_k x)(\log_7 k) = 3.
\]

Hence \( x = 7^3 = 343 \). Answer: (B).

35. Let us observe that \( 2^x \) is around 2004 if \( x = \frac{\ln 2004}{\ln 2} \approx 10.96 \). In fact \( 2^{11} - 4 \times 11 = 2004 \). So, one solution is \( x_1 = 11 \). We see that \(-4x = 2004\) implies \( x = -501 \) and \( 2^{-501} \) is almost 0 \((\approx 0.15 \times 10^{-150})\) which suggests that \( x_2 \approx -501 \). Hence the sum of the two solutions must be very close to \(-501 + 11 = -490\). Answer: (D).

36. The side lengths of this triangle must be 3a, 4a, 5a, and so its area is 6a² = 726. This equation gives \( a = 11 \) in and then the hypotenuse is 55 in. Answer: (D).

37. Let us observe that \( \sin x = 1 \) for \( x = \pi/2 + 2k\pi \) where \( k \) is an integer. So the intersection of \( \sin x \) and \( x/2004 \) is going to be around these values of \( x \) for \( x \) large (see figure). In fact we are going to have intersections around \( x_k = \pi/2 + 2k\pi \) as long as \( \frac{x_k}{2004} < 1 \). The greatest \( k \) with this property is 318. Using a calculator one can check that \( (2\pi \times 319 + \pi/4)/2004 > 1 \). This means that we cannot have any other solution of our equation larger than \( 2\pi \times 319 \). Then for each \( k \) from 0 to 318 there are two solutions around \( x_k \). Excluding 0 we get a total of \( 2 \times 319 - 1 = 637 \). Answer: (A).

38. The equation can be written as \( \frac{x - 3}{2} + \frac{x - 2}{3} = \frac{2}{x - 3} + \frac{3}{x - 2} \) or \( \frac{5x - 13}{6} = \frac{5x - 13}{(x - 3)(x - 2)} \). So, another solution of this equation is \( x_2 = \frac{13}{5} \) and solving \((x - 3)(x - 2) = 6\) one gets \( x_1 = 0 \) and \( x_3 = 5 \). Thus \( x_2x_3 = 13 \). Answer: (E).

39. If \( N = abcd ef \) with all digits different the number of possibilities would have been 6!. But if \( a = b \) the number of possibilities reduces to just a half because, for instance, \( fcadbe \) and \( fcbdae \) would count only as one permutation instead of two. For the same reasoning since \( c = d \) and \( e = f \) the number of all different permutation derived from \( N \) is \( 6!/2^3 = 90 \). Answer: (A).

40. Let us observe that the inequality makes sense only if \( 1 - x^3 \geq 0 \) which means \( x \leq 1 \) and \( 1 - \sqrt[3]{1 - x^3} \geq 0 \) which turns out to \( x \geq 0 \). So assuming that \( 0 \leq x \leq 1 \) we can safely
square both sides of the given inequality and obtain \( 1 - x^2 \geq \sqrt{1 - x^3} \). Squaring again we get equivalently \( x^2(x + 2)(x - 1) \geq 0 \). Since \( 0 \leq x \leq 1 \) this is possible if and only if \( x = 0 \) or \( 1 \). Answer: (B).

41. The graph of \( f(x) = x + 2|x + 1| + 2|x - 1| \) is shown in the accompanying figure. The problem is just another way of asking for the minimum of this function which is \( f(-1) = 3 \). Answer: (A).

42. Let \( G, H, I, \) and \( J \) be the projections of \( A, E, D \) and \( B \) on \( FC \). Then \( FG = JC = (FC - GJ)/2 = 12 \) in. From the right triangle \( BJC \) we obtain \( BJ = \sqrt{13^2 - 12^2} = 5 \). Similarly \( FH = IC = (FC - HI)/2 = 15 \) and \( ID = \sqrt{17^2 - 15^2} = 8 \). Then the area of \( ABDE \) is \( (10 + 4) \times (5 + 8)/2 = 91 \) in\(^2 \). Answer: E.

43. Let \( g(x) = (x^2 + x - 1)^{x-3} \) defined on integers. We observe that \( g(-2), g(-1), g(1) \) and \( g(3) \) are all equal to 1. On the other hand we get \( g(0) = -1 \) and \( g(2) = 1/5 \). If \( x \geq 4 \) we can be sure that \( g(x) > 1 \) because \( x^2 + x - 1 > 1 \) and \( x - 3 > 0 \). Similarly, if \( x \leq -3 \) then \( g(x) < 1 \) because \( x^2 + x - 1 > 1 \) and \( x - 3 < 0 \). Hence the answer is (E).
44. Triangles $AMN$ and $ACB$ are similar since $m\angle MAN = m\angle CAB$ and $m\angle AMN = m\angle ACB$. Hence $\frac{AM}{AC} = \frac{MN}{BC}$ which gives $MN = (18 \times 7)/21 = 6$ in. Answer: A.

45. Applying logarithms both sides of $a^{2b} = b^{3a}$ we obtain $2b \ln a = 3a \ln b$. Substituting $b = 5a$ in this equation we get $10a \ln a = 3a(\ln 5 + \ln a)$. Simplifying by $a(a > 0)$ and then solving for $a$ one arrives at $a = 5^{3/7}$. Answer: (C).

46. Let $MN = x$. Triangles $DMN$ and $DAB$ are similar. So $\frac{x}{2} = \frac{ND}{BD}$. From the similarity of triangles $BNQ$ and $BDC$ we get $\frac{2x}{6} = \frac{BN}{BD}$. Adding these two equalities together we obtain $x(1/2 + 1/3) = 1$ which gives $x = 6/5 = 1.2$ cm. Answer: C.

47. Method I: Let $f(t) = 3t^2 - t - 1$. This quadratic function has two different zeroes in the interval $(-1, 1)$ because $f(-1) = 3 > 0$, $f(0) = -1 < 0$ and $f(1) = 1 > 0$. These two different zeroes are not opposite to one another and so the quadratic equation in $\cos(2x) = 2\cos(x)^2 - 1$ has two different solutions. This means that the problem is determined uniquely because the free coefficient is $-1$. To determine the equation in $\cos 2x$ one can proceed as follows. From the given equation we have $3(1 + \cos 2x) - 2\cos x - 2 = 0$ or $2\cos x = 1 + 3 \cos 2x$. Squaring both sides one gets $4(\cos x)^2 = 2 + 2 \cos 2x = 9(\cos 2x)^2 + 6 \cos 2x + 1$ or $9(\cos 2x)^2 + 4 \cos 2x - 1 = 0$. Since the solution should be unique $a = 9$ and $b = 4$. Answer: D.

Method II: One can solve the quadratic $f(t) = 0$ and find the solutions $t_{1,2} = \frac{1 \pm \sqrt{13}}{6}$ then substitute in the double angle formula to obtain the solutions in $\cos 2x$: $\frac{-2 \pm \sqrt{13}}{9}$. Since their
product is \(-1/9\), \(a\) must be 9 and because their sum is \(-\frac{4}{3}\), \(b\) must be 4. Hence answer is (D).

48. Write \(f\) as \(f(x, y) = (x - 2)^2 + (y + 3)^2 - 13\). Using the distance formula between two points 
\[
d(x, y) := d((x, y), (2, -3)) = \sqrt{(x - 2)^2 + (y + 3)^2}
\]
we see that \(f(x, y) = d(x, y)^2 - 13\). So, to minimize \(f\) over the unit square is the same as minimizing the distance from a point in this square to the point of coordinates \((2, -3)\). The closest point in the square to \((2, -3)\) is then \((1, 0)\). So \(f(1, 0) = -3\) must be the minimum. Answer: C.

49. The slope of the tangent line is \(\frac{dy}{dx}|_{x=1} = 2\) so the equation of this tangent line is 
\(y - 1 = 2(x - 1)\) which means \(y = 2x - 1\). Therefore the \(y\)-intercept is \(-1\). Answer: E.

50. \[
\lim_{x \to 0} \frac{3^x - 1}{2^x - 1} = \lim_{x \to 0} \frac{\frac{3^x - 1}{x}}{\frac{2^x - 1}{x}} = \frac{d\frac{3^x}{dx}|_{x=0}}{d\frac{2^x}{dx}|_{x=0}} = \frac{3^0 \ln 3}{2^0 \ln 2} = \frac{\ln 3}{\ln 2} = \log_2 3.
\]
Answer: A.