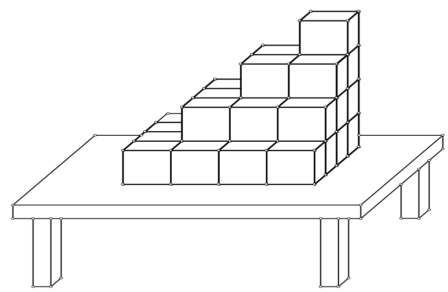
28^{th} Annual CSU Invitational Mathematics Tournament- Solutions

Sponsored by Columbus State University
Department of Mathematics
March 2, 2002

1. $1+2^2+3^2+4^2 = 1+4+9+16 = 30$,
   so the answer is (b)



2. $\sqrt{4} = 2$, $\sqrt[3]{27} = 3$, $2.71 = \dfrac{271}{100}$, $3.141592654 = 3141592654/10^9$, so the answer is (b).

3. $A = \dfrac{36B}{100}$, $C = \dfrac{40B}{100}$ and so $\dfrac{A}{C} = \dfrac{36}{40} = \dfrac{9}{10} = 0.9$ which means that (e) is the correct answer.

4. The prime factorization of 150 is $150 = 2 \times 3 \times 5^2$ which involves 2, 3 and 5 as prime factors. The answer is then (e).

5. $\frac{\textit{Area of a disk of radius } r}{\textit{Circumference of a circle of radius } r} = \frac{\pi r^2}{2\pi r} = \frac{r}{2}$ and since $r = 10$ the answer is (e).

6. Since $\dfrac{2n}{2n+11} = \dfrac{n}{n+5.5}$, $\dfrac{3n}{3n+17} = \dfrac{n}{n+5.\overline{6}}$, $\dfrac{4n}{4n+23} = \dfrac{n}{n+5.75}$ and $\dfrac{5n}{5n+29} = \dfrac{n}{n+5.8}$ and because fractions with smaller denominator are bigger the answer is (a).

7. We have $\dfrac{x+y}{x^2-x} \times \frac{x^3-x^2}{x^2-y^2} = \dfrac{x+y}{x(x-1)} \times \frac{x^2(x-1)}{(x-y)(x+y)} = \frac{x}{x-y}$, hence the answer is (b).

8. $2002 = 2 \times 7 \times 11 \times 13$ and $2 + 7 + 11 + 13 = 33$ and thus we need to choose (e).

9. $\left(\dfrac{4}{5}\right)^2 + \left(\dfrac{1}{3}\right)^2 = \dfrac{169}{225}$, and $\left(\dfrac{13}{15}\right)^2 = \dfrac{169}{225}$ which means by the converse of Pythagorean theorem that we have a right triangle: answer (c).

10. We can solve the given system of the simultaneous equations, or we can observe that $5x + 3y = (x - y) + 4(x + y) = (-3) + 4 \times 1 = 1$. Therefore the answer is (e).

11. The slope of the line through the points $(3, m)$ and $(m, 15)$ is $\dfrac{m-15}{3-m}$ and then the equation $\dfrac{m-15}{3-m} = m$ reduces to the quadratic equation $m^2 - 2m - 15 = 0$. Solving this equation we obtain $m_1 = -3$ and $m_2 = 5$ as its solutions. Thus the answer is (b).

12. $f_6 = f_5 + f_4 = 5 + 3 = 8$, $f_7 = f_6 + f_5 = 8 + 5 = 13$, $f_8 = f_7 + f_6 = 13 + 8 = 21$, $f_9 = f_8 + f_7 = 21 + 13 = 34$ and finally $f_{10} = f_9 + f_8 = 34 + 21 = 55$ which means we need to choose (b)

13. The missing rows of the *Pascal's Triangle* are:

$$\begin{array}{ccccccccccccccccc}
 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \\
1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1
\end{array}$$

in which see that the 9^{th} row contains only two odd entries. Thus (a) is the correct choice.

14. We need to solve the system of simultaneous equations $x + y = 1750$ and $\dfrac{x}{y} = \dfrac{3}{4}$ which can be solved easily by inspection: $x = \dfrac{3}{7} \times 1750 = 750$ and $y = \dfrac{4}{7} \times 1750 = 1000$. Hence $y - x = 250$ and the answer is (e)

15. Let x be the common solution. Then $2x^2 - 5x + 2 = 0$ and $2x^2 - 7x + 3 = 0$ which by subtraction gives $2x - 1 = 0$. Hence the common solution is $\dfrac{1}{2}$. Since the product of

the roots of the first quadratic equation is 1 the other root is 2 and similarly the other root of the second quadratic equation is 3. The sum of these two solutions is then 5: answer (a).

16. $\log_2(\log_2 x) = 2 \Rightarrow \log_2 x = 2^2 \Rightarrow x = 2^4 = 16$. Answer=(a).

17. Clearly $m(\angle B) + m(\angle C) = 120°$. As in the Problem # 14, $m(\angle B) = \frac{2}{5} \times 120° = 48°$ and $m(\angle C) = \frac{3}{5} \times 120° = 72°$. Then $m(\angle AIB) = 180° - m(\angle A)/2 - m(\angle B)/2 = 126°$. Thus the correct answer is (e).

18. Let x be the average of the adults. The weight of all the people can be expressed in two ways: $(35 + 44 + 41) \times 91 = 35 \times 60 + 44 \times 70 + 41 \times x)$. Solving for x we get $x = 140$. Then the answer is (b).

19. Using the distance formula the property of P translates into $\sqrt{(x-8)^2 + y^2} = 2\sqrt{(x-2)^2 + y^2}$ or $x^2 - 16x + 64 + y^2 = 4(x^2 - 4x + 4 + y^2)$. After reducing the terms alike we obtain $x^2 + y^2 - 16 = 0$ which shows that (b) is the correct answer.

20. Let as denote by $\mathcal{M}7$ an integer multiple of 7. Then $2001 = 2002 - 1 = 286 \times 7 - 1 = \mathcal{M}7 - 1$. Then it easy to observe (using binomial formula) that $2001^{even\ power} = \mathcal{M}7 + 1$ and $2001^{odd\ power} = \mathcal{M}7 - 1$. Therefore the correct answer is (b).

21. Let x be the number of Martians and y be the number of Jovians. From the information given we have $2x + 3y = 51$ and $3x + 5y = 81$. We can solve this system of simultaneous equations or observe that $x - y = 8(2x + 3y) - 5(3x + 5y) = 8 \times 51 - 5 \times 81 = 3$. Thus the answer is (a).

22. Denote by x the greatest of the integers. Then x satisfies the equation $x + (x-1) + ... + (x - 90) = 2002$. Using the formula of summation $1 + 2 + 3 + ... + n = \dfrac{n(n+1)}{2}$, $n \in \mathbb{N}$, the equation becomes $91x - \dfrac{90 \times 91}{2} = 2002$. Equivalently, after dividing by 91 both sides we get $x - 45 = 22$ which implies $x = 67$ and so the correct answer is (e)

23. Let n be a counting number in the hundreds and let us find the number of digits in the writing $12345678910111213...(n-1)n$. Since each counting number between 10 and 99 contributes with 2 digits and every counting number between 100 and n contributes with 3 digits we obtain $9 + 2(99 - 10 + 1) + 3(n - 100 + 1) = 3n - 108$ digits. Since

we have gotten a multimple of three as the number of digits let us solve the equation $3n-108=2001$. This yields $n=703$. Therefore the 2002^{nd} digit is 7.

24. A calculation shows that $g(g(x)) = -\frac{1}{x}$. Hence $g(g(g(g(x)))) = -\frac{1}{-\frac{1}{x}} = x$ and so the answer is (e).

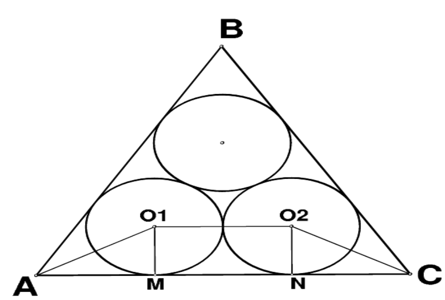
25. The given equation implies $x+5 = x-4$ or $x+5 = -(x-4)$. We get only one solution which is $x = -\frac{1}{2}$. Using this informatiion we can exclude (a), (b), (c) and (d) as being the correct answer. Observe that (e) is in fact an equivalent way of writing the given equation since $|a|^2 = a^2$ for every real number a and $u = v \Leftrightarrow u^2 = v^2$ for every two nonnegative real numbers u, v.

26. The graph of the given equation is a rhombus with vertices at $(-80, 4)$, $(74, 4)$, $(-3, -22)$ and $(-3, 30)$. It is easily seen that the area of a rhombus is equal to half the product of its diagonals. This leads us to $Area = \frac{154 \times 52}{2}$ and so the answer is (a).

27. Let $\frac{2x+1}{3x+2} = y$ and solving for x one gets $x = \frac{2y-1}{2-3y}$. This says that (b) is the correct answer.

28. One of the algebra theorems says that the remainder of a polynomial $P(x)$ divided by $x-a$ is $P(a)$. Since $P(-1) = (-1)^3 - 2(-1)^2 + 3(-1) - 4 = -10$ the answer is (c)

29. Let r be the radius of these circles (in inches). In the accompanying figure clearly $MN = 2r$ and $AM = NC = r\sqrt{3}$ since $\angle O_1AM = \angle O_2CN = 30°$. Then r satisfies the condition $2r\sqrt{3} + 2r = 4$. Solving for r and rationalizing we get $r = \sqrt{3} - 1$. Answer=(a).



30. Squaring both sides of the given equation we get $2\sqrt{11x-2} = x+10$. Hence if $x \geq -10$ we can square both sides again to obtain the equivalent equation $x^2 - 24x + 108 = 0$. Because $\Delta = b^2 - 4ac = 24^2 - 4 \times 108 = 144 \geq 0$ we have two real positive solutions whose sum is then $x_1 + x_2 = -\frac{b}{a} = 24$. Answer:=(c).

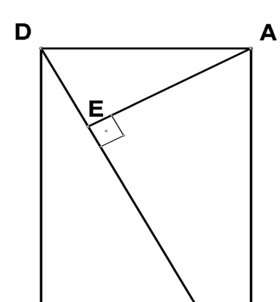
31. Let a and b be the numbers turned up on the two dice. Let us find the probability that ab is not a multiple of 3. The product ab is not divisible by 3 if and only if $a, b \in \{1, 2, 4, 5\}$. This gives $4 \times 4 = 16$ possible choices for (a, b) out of a toatal of 36. So, the probability that ab is not divisible by 3 is $\dfrac{16}{36} = \dfrac{4}{9}$ and thus the required probability is $1 - \dfrac{4}{9} = \dfrac{5}{9}$. Answer:=(a)

32. Let $a \leq b \leq c$ be three positive integers such that $a^4 + b^4 + c^4 = 2002$. Since $3 \times 5^4 = 1975 < 2002 < 7^4 = 2401$ it follows that $c = 6$. Then $a^4 + b^4 = 2002 - 6^4 = 706$. Similarly, $2 \times 4^4 = 512 < 706 < 6^4 = 1296$ implies that $b = 5$ and finally $a = \sqrt[4]{706 - 5^4} = 3$. Adding up a, b and c we see that the correct answer is (d).

33. Denote by x the number of gallons of 98% concentrate which is mixed with $(12 - x)$ gallons of 50% concentration. Since the new mixture has a 70% concentration x must satisfy the linear equation $\dfrac{98x}{100} + \dfrac{50(12-x)}{100} = \dfrac{70 \times 12}{100}$ which has the solution $x = 5$. Answer:=(d).

34. Set $f(x) = y$ and solve for x. We obtain $x = \dfrac{3y-1}{2(y+3)}$ which is well defined for every real number y except $y = -3$. Therefore the range of f is $\mathbb{R} \setminus \{-3\}$. Answer:=(a)

35. Triangles ADE and BAE are similar by the (AA-i.e. angle-angle) case of similarity. The propotionality of their sides implies $\dfrac{AE}{BE} = \dfrac{DE}{AE}$ which is equivalent to $AE^2 = BE \times DE$. Hence $AE = \sqrt{12 \times 3} = 6$ and then the area of the rectangle is $2 \times \dfrac{(3+12) \times 6}{2} = 90\ in^2$. Answer:=(b).



36. Since α is an angle in the third quadrant $\sin\alpha = -\sqrt{1-\cos^2\alpha} = -\dfrac{12}{13}$. Using the double angle formulae we get $\tan(2\alpha) = \dfrac{\sin(2\alpha)}{\cos(2\alpha)} = \dfrac{2\sin\alpha\cos\alpha}{\cos^2\alpha - \sin^2\alpha} = \dfrac{2\times 5\times 12}{5^2-12^2} = -\dfrac{120}{119} \approx -1.0084$. Answer:=(b).

37. **Method 1.** Set $x+y=m$, solve for y and substitute in the inequality given to obtain $2x^2+(4-2m)x+m^2-2m \le 0$. Clearly this inequality has at least one solution in x if and only if $\Delta = b^2-4ac = (4-2m)^2 - 4\times 2\times(m^2-2m) \ge 0$. Simplifying we obtain the equivalent inequality $16-4m^2 \ge 0$ or $m \in [-2,2]$. Answer:=(e).

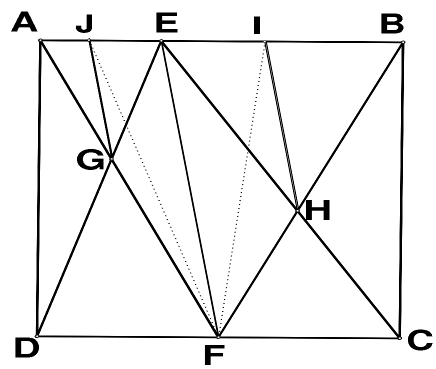
    Method 2. Observe that $(a+b)^2 \le 2(a^2+b^2)$ for every real numbers a and b. Then $(x+y)^2 = (x+1+y-1)^2 \le 2\left((x+1)^2+(y-1)^2\right) \le 2\times 2 = 4$ which implies the same conclusion as above: $x+y \in [-2,2]$.

38. The tangent line to the circle at the point $\left(-\dfrac{3}{5}, \dfrac{4}{5}\right)$ is perpendicular to the radius corresponding to this point. This radius has slope $\dfrac{\frac{4}{5}}{-\frac{3}{5}} = -\dfrac{4}{3}$ and so the tangent line has slope $-\dfrac{1}{-\frac{4}{3}} = \dfrac{3}{4}$. The equation of this tangent line is then $y - \dfrac{4}{5} = \dfrac{3}{4}(x + \dfrac{3}{5})$

which for $x = 0$ yield $y = \frac{5}{4}$. Answer:=(c).

39. **Method 1.** Let $AB = a$ and $BC = b$ (in meters). We know that $ab = 70\ m^2$. Let us observe that $Area(DEC) = ab/2 = 35\ m^2$ and $Area(ADF) = \frac{1}{2}\frac{a}{2}b = \frac{35}{2}\ m^2$. Clearly, $AE = \frac{a}{3}$, $EB = \frac{2a}{3}$ and $DF = FC = \frac{a}{2}$. Because of obvious similarity of the triangles DGF and EGA we have $\frac{GF}{GA} = \frac{DF}{AE} = \frac{3}{2}$. Hence $\frac{Area(DGF)}{Area(ADF)} = \frac{GF}{AF} = \frac{3}{5}$. Therefore $Area(DGF) = \frac{3}{5} \times \frac{35}{2} = \frac{21}{2}\ m^2$. Similarly $Area(FHC) = \frac{15}{2}\ m^2$ and then $Area(EGFH) = 35 - \frac{21}{2} - \frac{15}{2} = 17\ m^2$.Answer=(e).

Method 2. In the accompanying figure J and I are situated on $\overline{AB}$ such that $\overline{GJ}$ and $\overline{HI}$ are parallel to EF. It is easy to observe that the quadrilateral EGFH has the same area as the triangle JFI. Then $Area(JFI) = \frac{JI \times b}{2} = \frac{JI}{a}\frac{ab}{2} = 35 \times \frac{JI}{a}$.

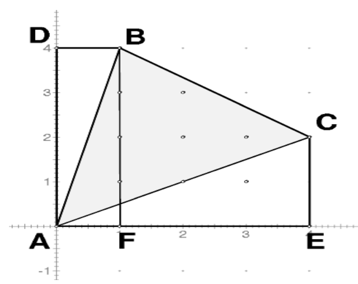


We now calculate JI in terms of a. Because $\overline{GJ}$ is parallel to $\overline{EF}$ and from the similarity of the triangles AGE and FGD we have $\frac{JE}{JA} = \frac{GF}{AG} = \frac{DF}{AE} = \frac{3}{2}$ which in turn implies that $JE = \frac{3}{5}AE = \frac{a}{5}$. Similarly $EI = \frac{2a}{7}$. Then $JI = \frac{a}{5} + \frac{2a}{7} = \frac{17a}{35}$. Therefore $Area(EGFH) = 35 \times \frac{JI}{a} = 17\ m^2$.

40. Since $3f(x) - 2 = \frac{8}{3x-1}$ if $f(x)$ is an integer so is $3f(x) - 2$. Thus $3x - 1$ must be an integer divisor of 8: -8, -4, -2, -1, 1, 2, 4 or 8. On the other hand only -4, -1, 2 and 8 are of the form $3x - 1$ for x equal to -1, 0, 1, and 3. Because $f(-1)$, $f(0)$, $f(1)$ and $f(3)$ are indeed integers we see that the right choice is (c).

41. $x^3+y^3=(x+y)(x^2-xy+y^2)=1\times\left(\frac{3(x^2+y^2)}{2}-\frac{(x+y)^2}{2}\right)=\frac{9}{2}-\frac{1}{2}=4$. Answer=(e).

42. **Method 1.** One can use Pick's theorem to solve this problem. Call all the points of integer coordinates lattice points. According to Pick's theorem $Area(ABC)=\frac{b}{2}+i-1$ where b is number of lattice points on the sides of the triangle ABC and i is the number of lattice points in the interior of ABC. Then $Area(ABC)=\frac{4}{2}+6-1=7$. Answer=(b).
Method 2. Alternatively, let D, E and F as in the accompanying figure. Then $Area(ABC)=Area(ADBF)+Area(FBCF)-Are(ADB)-Area(AEC)=4+\frac{3(4+2)}{2}-\frac{4\times1}{2}-\frac{4\times2}{2}=7$.

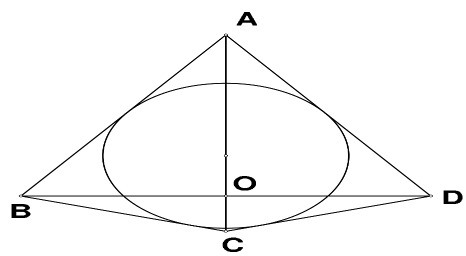


43. Since $a^2=(x_1+1)+(x_2+1)=(x_1+x_2)+2=-a+2$ we need to have $a=1$ or $a=-2$. The value 1 for a is not convenient because the equation $x^2+x+1=0$ has no real solutions. Answer=(c).

44. Substituting $x=\frac{5}{3}$ in the functional equation given we get $f\left(\frac{5}{3}\right)+\frac{5}{3}f\left(-\frac{2}{3}\right)=\frac{14}{3}$. Substituting $x=-\frac{2}{3}$ in the same equation we obtain $f\left(-\frac{2}{3}\right)-\frac{2}{3}f\left(\frac{5}{3}\right)=\frac{7}{3}$. Solving this system of simultaneous equations in $f\left(\frac{5}{3}\right)$ and $f\left(-\frac{2}{3}\right)$ one gets $f\left(\frac{5}{3}\right)=\frac{7}{19}$.

Answer=(c).

45. Using the Pythagorean theorem we get $BC = DC = \sqrt{24^2 + 7^2} = 25$ and $AD = AB = \sqrt{24^2 + 32^2} = 40$. The area of quadrilateral ABCD is $2 \times (32 + 7) \times 24/2 = 936$. Let r be the radius of the inscribed circle and I be its center. Then $Area(ABCD) = Area(AIB) + Area(BIC) + Area(CID) + Area(DIA) = \frac{AB \times r}{2} + \frac{BC \times r}{2} + \frac{DC \times r}{2} + \frac{AD \times r}{2} = r(25 + 40)$. Thus $r = \frac{936}{65} = \frac{72}{5} = 14.4$ which points that (b) is the correct answer.



46. Using the change of base formula and denoting $\log_3 x = t$ we have $\frac{t}{2} + \frac{1}{2t} = 1$. Solving for t we get $t = 1$ and so $x = 3$. Answer=(d).

47. Because α is an angle in the first quadrant $\cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}} = \frac{20}{\sqrt{20^2+21^2}} = \frac{20}{29}$. Answer=(c).

48. We have $(1+j)^2 = 1 + 2j + j^2 = 2j$ and so $(1+2j)^{2n} = 2^n j^n$. Since $j^{2001} = j^{2000} j = (j^4)^{500} j = j$ we see that (c) is the correct answer

49. Solving for m we obtain $m = \frac{5n}{n-4}$. Then $m - 5 = \frac{20}{n-4}$. Thus $n - 4$ must be an integer divisor of 20 greater that -3. So, $n - 4 \in \{20, 10, 5, 4, 2, 1, -1, -2\}$ or $n \in \{24, 14, 9, 8, 6, 5, 3, 2\}$. But for $n = 3$ or $n = 2$ the corresponding value of m is negative. This leaves us with only 6 ordered pairs satisfying the given equation.

Answer=(e).

50. Observe that the area of the triangle ACD is the same as the area of the triangle CDO. where O is the center of the circle of diameter $\overline{AB}$. The triangle CDO is obviously equilateral (all angles congruent). Its area is then $\dfrac{6^2 \times \sqrt{3}}{4} = 9\sqrt{3}$. Answer=(d).