Forty Second Annual Columbus State University Invitational Mathematics Tournament

Sponsored by
The Columbus State University
Department of Mathematics
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The Columbus State University Mathematics faculty welcomes you to this year's tournament and to our campus. We wish you success on this test and in your future studies.

Introduction

This is a 90-minute, 50-problem, multiple-choice exam. There are five possible responses to each question. You should select the one best answer for each problem. In some instances this may be the closest approximation rather than an exact answer. You may mark on the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer circle the choice you have made on the test booklet. Carefully transfer your answers to the score sheet. Completely darken the blank corresponding to the letter of your response to each question. Mark your answer boldly with No.2 pencil. If you must change an answer, completely erase the previous choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response. The examination will be scored on the basis of $+12$ for each correct answer, -3 for each incorrect selection, and 0 for each omitted item. Each student will be given an initial score of $+200$.

Pre-selected problems will be used as tie-breakers for individual awards. These problems, designated with an asterisk ($*$), in order of consideration are: 4, 10, 14, 16, 17, 21, 24, 28, 31, 36, 40, 41, 42, 43, 45, 46, 47 and 50.

Throughout the exam, $\overline{AB}$ will denote the line segment from point A to point B and AB will denote the length of $\overline{AB}$; $\angle A$ denotes the $\angle BAC$ or $\angle CAB$ in the triangle $\triangle ABC$. Pre-drawn geometric figures are not necessarily drawn to scale.

Review and check your score sheet carefully. **Your student identification number and your school number must be encoded correctly on your score sheet.**

When you complete your test, bring your pencil, scratch paper and answer sheet to the test monitor. Leave the room after you have handed in your answer sheet. Please leave quietly so as not to disturb the other contestants. Do not congregate outside the doors by the testing area. You may keep your copy of the test. Your sponsor will have a copy of solutions to the test problems.

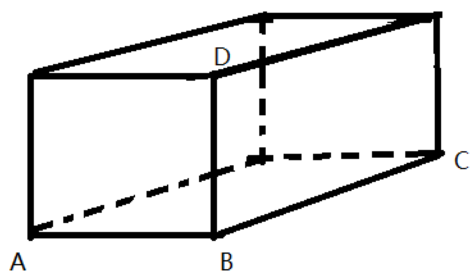
Do not open your test until instructed to do so!

1. If the straight lines $2y + 3x - 4 = 0$ and $3y - ax + 4 = 0$ are perpendicular to each other, what is the value of a?

   $A)$ 5 $B)$ 4 $C)$ 3 $D)$ 2 $E)$ 1

   Solution: The slopes of these two lines are $\frac{-3}{2}$ or $\frac{a}{3}$. The product of their slopes is equal to -1. We have $a = 2$. The correct answer is D.

2. In the cuboid represented below, we know $AB = BD = 1$ and $BC = 2$. What is the total surface area of the cuboid?



   $A)$ 6 $B)$ 2 $C)$ 8 $D)$ 9 $E)$ 10

   Solution: The total surface area equal to $2(AB \cdot BC + AB \cdot BD + BD \cdot BC) = 10$. The correct answer is E.

3. Let i be the complex number $\sqrt{-1}$. Find the value of i^{2016}.

   $A)$ i $B)$ $-i$ $C)$ 1 $D)$ -1 $E)$ 0

   Solution: Since

$$i^{2016} = (i^2)^{1008} = (-1)^{1008} = 1.$$

   The correct answer is C.

4. *Find the minimum value of the function $f(x) = \frac{x+1}{x}$ for real numbers x, where $1 \leqslant x \leqslant 2016$.

   $A)$ 1 $B)$ $\frac{3}{2}$ $C)$ $\frac{2017}{2016}$ $D)$ 2 $E)$ $\frac{2016}{2015}$

   Solution: Since $f(x) = \frac{x+1}{x} = 1 + \frac{1}{x}$ is a decreasing function of x, the minimum value of the function occurs at $x = 2016$. The correct answer is C.

5. The average age of three girls is 7 years. If a boy joins the group, then the average age of these four children is 6 years. What is the boy's age?

$A)$ 1 $B)$ 2 $C)$ 3 $D)$ 4 $E)$ 5

Solution: Denote the ages of the girls by a, b, and c and the age of the boy by t. Then we have $a+b+c=21$ and $a+b+c+t=24$, so $t=3$. The correct answer is C.

6. For what value of k does the system of equations

$$\begin{cases} y = x^2+5 \\ y = 2x+k \end{cases}$$

have a unique solution?

$A)$ 1 $B)$ 2 $C)$ 3 $D)$ 4 $E)$ 5

Solution: The system of equations is equivalent to the equation $x^2-2x+5-k=0$ which has a unique solution if and only if the discriminant $4+4k-20=0$. It follows that $k=4$. The correct answer is D.

7. Find the radius of the circle $x^2-2x+y^2-4y-4=0$.

$A)$ 1 $B)$ 2 $C)$ 3 $D)$ 4 $E)$ 5

Solution: Complete the squares to get the equation in the form $(x-1)^2+(y-2)^2=3^2$. The correct answer is C.

8. Find the vertex of the parabola $y=x^2-2x+3$.

$A)$ $(1,2)$ $B)$ $(1,-2)$ $C)$ $(-1,2)$ $D)$ $(0,3)$ $E)$ $(2,3)$

Solution: $y=x^2-2x+3 \Rightarrow (y-2)=(x-1)^2$. The correct answer is A.

9. What is the product of all solutions of the equation $x^4-2x^3+4x^2+6x+3=0$?

$A)$ 1 $B)$ 2 $C)$ 3 $D)$ 4 $E)$ 5

Solution: Let a, b, c, and d be the solutions of the equation. Then $x^4-2x^3+4x^2+6x+3=(x-a)(x-b)(x-c)(x-d)$, so the product $abcd=3$. The correct answer is C.

10. *If the geometric mean of two positive real numbers a and b (with $a < b$) is equal to 15 and their arithmetic average is 39, what is $\sqrt{b/a}$?

$A)$ 1 $B)$ 2 $C)$ 3 $D)$ 4 $E)$ 5

Solution: The problem reduces easily to the quadratic equation $x^2 - 78x + 225 = 0$ and so we obtain $a = 3$ and $b = 75$. The correct answer is E.

11. If x is an integer, what is the minimum value of $(2x - 3)^2 + 2$?

$A)$ 1 $B)$ 2 $C)$ 3 $D)$ 4 $E)$ 5

Solution: Since x is an integer, then $(2x - 3)^2 \geq 1 \Rightarrow (2x - 3)^2 + 2 \geq 3$. Note that $(2x - 3)^2 + 2 = 3$ for $x = 1$ or $x = 2$. The correct answer is C.

12. There are 10 people in a room, 40% percent of whom are men. If no man enters or leaves the room, how many women must enter the room so that 20% of the total number of people in the room are men?

$A)$ 10 $B)$ 9 $C)$ 3 $D)$ 7 $E)$ 6

Solution: Note that 4 is 40% of 10. There are 4 men in the room. Let x be the number of women entering the room. Then the percentage of men in the room is given by $\frac{4}{10 + x} \cdot 100 = 20$. Solving for x yields $x = 10$. The correct answer is A.

13. If $(1 - 2x)^{2016} = a_0 + a_1x + a_2x^2 + \cdots + a_{2016}x^{2016}$, find the value of $a_0 + a_1 + a_2 + \cdots + a_{2016}$.

$A)$ 5 $B)$ 4 $C)$ 3 $D)$ 1 $E)$ 2

Solution: Let $x = 1$, we have $(1 - 2)^{2016} = a_0 + a_1 + a_2 + \cdots + a_{2016} = 1$. The correct answer is D.

14. *Assume that the sets $\left\{1, a+b, a\right\}$ and $\left\{0, \frac{b}{a}, b\right\}$ have the same elements for real numbers a and b. Find the value of $b - a$.

$A)$ 1 $B)$ 2 $C)$ 3 $D)$ 4 $E)$ 0

Solution: Since $0 \in \{1, a + b, a\}$ and $a \neq 0$, we have $a + b = 0$. Hence $\frac{b}{a} = -1$. Note that $-1 \in \{1, a + b, a\}$, we have $a = -1$, $b = 1$.

The correct answer is B.

15. If $(x - 3x^2)^3 = a_0 + a_1x + a_2x^2 + \cdots + a_5x^5 + a_6x^6$, find the value of $a_0 + a_1 - a_2$.

$A)$ 3 $B)$ 4 $C)$ 0 $D)$ 1 $E)$ 2

Solution: Expanding gives

$$(x - 3x^2)^3 = x^3(1 - 3x)^3 = x^3(1 - 9x + 27x^2 - 27x^3).$$

The correct answer is C.

16. *How many real solutions does the equation $(x^2-3)^{x^2-2x}=1$ have?

$A)$ 1 $B)$ 2 $C)$ 3 $D)$ 4 $E)$ 5

Solution: We should consider the case when the base equals 1 and the case when the base is not equal to 1.

$$\text{Case 1}: x^2-3=1 \Rightarrow x=\pm 2.$$
$$\text{Case 2}: x^2-2x=0 \text{ and } x^2-3\neq 0 \Rightarrow x=0,2.$$

The correct answer is C.

17. *Each 3-digit positive integer is written on a card. All cards are placed in a box and one is extracted from the box. Find the probability that the sum of the digits on the card is 5.

$A)\ \frac{1}{900}$ $B)\ \frac{1}{300}$ $C)\ \frac{1}{150}$ $D)\ \frac{1}{90}$ $E)\ \frac{1}{60}$

Solution: The total number of 3-digit positive integers is 900. Each 3-digit positive integer with the sum of its digits equal to 5 is listed in the set below.

$$\{104, 113, 122, 131, 140, 203, 212, 221, 230, 302, 311, 320, 401, 410, 500\}.$$

The correct answer is E.

18. Let x and y be real numbers such that $x^2+10x+4y^2-20y+50=0$. Find the value of $-2(x+y)$.

$A)$ 1 $B)$ 2 $C)$ 3 $D)$ 4 $E)$ 5

Solution: Since

$$x^2+10x+4y^2-20y+50=(x+5)^2+(2y-5)^2=0 \Rightarrow x+y=\frac{-5}{2},$$

the correct answer is E.

19. If the real numbers x and y satisfy the system

$$\begin{cases} 3x+y &= k+1 \\ x+3y &= 3 \end{cases}$$

for the real number k with $2<k<4$, which one of the following is true about $x-y$?

$A)\ 0<x-y<1/2$ $B)\ 0<x-y<1$ $C)\ -3<x-y<-1$

$D)\ -1<x-y<1$ $E)\ 3<x-y<5$

Solution: Subtracting the second equation from the first, we have $x-y=\frac{k}{2}-1$. Since $2<k<4$, $0<\frac{k}{2}-1<1$. The correct answer is B.

20. Simplify the expression $\dfrac{4\log\sqrt{b}\cdot\log_b a}{\log a}$ for $a>1$ and $b>1$.

A) 1 *B*) 2 *C*) 3 *D*) 4 *E*) 5

Solution: Note that

$$\frac{2\log b\cdot\log_b a}{\log a}=\frac{2(\log b)\frac{\log a}{\log b}}{\log a}=2.$$

The correct answer is B.

21. *If $f(x)$ is an odd function defined on $(-\infty,\infty)$ such that $f(x)=f(1-x)$, what is the value of $f(2016)$?

A) 0 *B*) 1 *C*) 2 *D*) 3 *E*) 4

Solution: Since

$$\begin{gathered}f(0)=0=f(1)=f(-1),\\ f(x+1)=f(-x)=-f(x)\Rightarrow f(x+2)=f(-x-1)=f(x).\end{gathered}$$

This means the period of function $f(x)$ is 2. $f(2016)=f(0+1008\cdot 2)=f(0)=0$.

The correct answer is A.

22. Find the number of distinct pairs (x,y) of positive integers which are solutions of the equation $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{7}$.

A) 1 *B*) 2 *C*) 3 *D*) 4 *E*) 5

Solution: Since $xy=7(x+y)$, we have $(x-7)(y-7)=49$. Using the factors of 49, there are only three possibilities ($49=7\cdot 7$, $49=1\cdot 49$, and $49=49\cdot 1$).

The correct answer is C.

23. Find all the values of a for which $B=\{x|x^2-ax-4\leqslant 0\}$ contains the interval $[2,4]$.

A) $[-1,2)$ *B*) $[-1,2]$ *C*) $[3,\infty)$ *D*) $[0,3]$ *E*) $[2,\infty)$

Solution: Denote $A=[2,4]$.

$$x^2-ax-4\leqslant 0\Rightarrow\frac{a-\sqrt{a^2+16}}{2}\leqslant x\leqslant\frac{a+\sqrt{a^2+16}}{2}.$$

Note that $\dfrac{a-\sqrt{a^2+16}}{2}<0$ and $\dfrac{a+\sqrt{a^2+16}}{2}>0$ for any real number a. $A\subseteq B$ if and only if $4\leqslant\dfrac{a+\sqrt{a^2+16}}{2}$. That is $8-a\leqslant\sqrt{a^2+16}$, which means $a\geqslant 3$. Therefore, The correct answer is C.

24. *Consider the function $f(x) = \dfrac{5-4x+x^2}{2-x}$ for $-\infty < x < 2$. Find the minimum value of $f(x)$.

A) 0 *B*) 1 *C*) 2 *D*) 3 *E*) 4

Solution: Clearly, $2-x>0$. Rewrite

$$f(x) = \frac{1}{2-x} + 2 - x \geqslant 2\sqrt{\frac{1}{2-x}(2-x)}.$$

$f(x)=2$ if and only if $\dfrac{1}{2-x} = 2-x$. We have $x = 1 \in (-\infty, 2)$. The correct answer is C.

25. If x and y satisfy the following system of equations, find the value of xy.

$$\begin{cases} \log x - \log y = -2 \\ \log x + 2\log y = 1 \end{cases}$$

A) 1 *B*) 2 *C*) 3 *D*) 4 *E*) 5

Solution: Note that $\log x - \log y = -2 \Rightarrow \dfrac{x}{y} = 10^{-2}$ and $\log x + 2\log y = 1 \Rightarrow xy^2 = 10$. Hence

$$xy^2/\frac{x}{y} = 10/10^{-2}.$$

This means $y = 10$ and $x = 10^{-1} = 0.1$. Therefore, $xy = 10 \cdot 10^{-1} = 1$. The correct answer is A.

26. Find the value of a in $x^3 - 3x + a = 0$ if the sum of two of its solutions is 2.

A) 1 *B*) 2 *C*) 3 *D*) 4 *E*) 5

Solution: Without loss of generality, we assume x_1, x_2, and x_3 are solutions of $x^3 - 3x + a = 0$, and $x_1 + x_2 = 2$. Since $x_1 + x_2 + x_3 = 0$, we have $x_3 = -2$ satisfying the equation. That is $(-2)^3 - 3\cdot(-2) + a = 0$, $a = 2$. The correct answer is B.

27. If x and b are real numbers such that $\log_{x^2} b + \log_{b^2} x = 1$, find the value of $\dfrac{x}{b}$.

A) 5 *B*) 4 *C*) 1 *D*) 2 *E*) 3

Solution: By the change of base formula, we have $\dfrac{\log b}{2\log x} + \dfrac{\log x}{2\log b} = \dfrac{(\log b)^2 + (\log x)^2}{2\log b \log x} = 1$. Rearrangement gives $(\log b - \log x)^2 = 0$ which implies $x = b$. The correct answer is C.

28. *If $3^{1-2x} = \pi$, find the value of 9^{1+x}.

$A)\ \dfrac{27}{\pi}$ $\qquad B)\ \dfrac{3}{\pi}$ $\qquad C)\ \dfrac{9}{\pi}$ $\qquad D)\ \dfrac{12}{\pi}$ $\qquad E)\ \dfrac{2}{\pi}$

Solution: From $3^{1-2x} = \pi$, we have $3^{2x} = 9^x = \dfrac{3}{\pi}$. Consequently

$$9^{1+x} = 9 \cdot 9^x = 9 \cdot 3^{2x} = \frac{27}{\pi}.$$

The correct answer is A.

29. Let $x = \sqrt{3+\sqrt{5}} - \sqrt{3-\sqrt{5}}$. Find the value of $\left(x^3 + 2x^2 - 2x - 5\right)^{2016}$.

$A)\ 5$ $\qquad B)\ 4$ $\qquad C)\ 3$ $\qquad D)\ 2$ $\qquad E)\ 1$

Solution: Since

$$\begin{aligned} &x^2 = 3+\sqrt{5}+3-\sqrt{5}-2\sqrt{3+\sqrt{5}}\sqrt{3-\sqrt{5}} = 6-2\sqrt{9-5} = 2, \\ &x^3+2x^2-2x-5 = x(x^2-2)+2(x^2-2)-1 = -1, \end{aligned}$$

the correct answer is E.

30. Find the value of the product $\left(1+\dfrac{1}{1}\right)\left(1+\dfrac{1}{2}\right)\left(1+\dfrac{1}{3}\right)\left(1+\dfrac{1}{4}\right)\cdots\left(1+\dfrac{1}{2016}\right)$.

$A)\ 2013$ $\qquad B)\ 2014$ $\qquad C)\ 2015$ $\qquad D)\ 2016$ $\qquad E)\ 2017$

Solution: The product is equal to

$$\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdot\dots\cdot\frac{2017}{2016} = 2017.$$

The correct answer is E.

31. *How many triples (x, y, z) are solutions of the equation $x^2+y^2+z^2+\dfrac{1}{x^2}+\dfrac{1}{y^2}+\dfrac{1}{z^2} = 6$.

$A)\ 2$ $\qquad B)\ 3$ $\qquad C)\ 4$ $\qquad D)\ 8$ $\qquad E)\ 6$

Solution: Using $x^2 + \dfrac{1}{x^2} - 2 = \left(x - \dfrac{1}{x}\right)^2 \geqslant 0$, the original equation can be rewritten as

$$\left(x-\frac{1}{x}\right)^2 + \left(y-\frac{1}{y}\right)^2 + \left(z-\frac{1}{z}\right)^2 = 0$$

This implies $x = \dfrac{1}{x}$, $y = \dfrac{1}{y}$, and $z = \dfrac{1}{z}$. Thus $x = \pm 1$, $y = \pm 1$, $z = \pm 1$ and they are independent. In total there are 8 possibilities. The correct answer is D.

32. Determine the number of real solutions of $x^2 - 3x - a^2 + 2 = 0$ for a real number a.

$A)$ 0 $B)$ 1 $C)$ 2 $D)$ 3 $E)$ 4

Solution: Since

$$x^2 - 3x - a^2 + 2 = \left(x - \frac{3 + \sqrt{4a^2+1}}{2}\right)\left(x - \frac{3 - \sqrt{4a^2+1}}{2}\right),$$

there are two real solutions for any given value of a. The correct answer is C.

33. Find the interval on which the function $f(x) = \log_{\frac{1}{2}}(x^2 - 2x - 3)$ is increasing.

$A)$ $(-\infty, +\infty)$ $B)$ $(-\infty, -1)$ $C)$ $(-1, 3)$ $D)$ $(3, +\infty)$ $E)$ $\big[3, +\infty\big)$

Solution: Denote $g(x) = x^2 - 2x - 3$. From $g(x) > 0$, we have $x < -1$ or $x > 3$. The function $g(x)$ is decreasing in $(-\infty, -1)$ and increasing in $(3, -\infty)$. The function $f(x)$ is a decreasing function. Hence $f(x)$ is increasing in $(-\infty, -1)$. The correct answer is B.

34. Find the value of $x + y$ such that x and y are real numbers which satisfy the following system of equations.

$$\begin{cases} (x-1)^3 + 2016(x-1) = -1 \\ (y-1)^3 + 2016(y-1) = 1 \end{cases}$$

$A)$ 1 $B)$ 2 $C)$ 3 $D)$ 4 $E)$ 5

Solution: From the original system, we have

$$\begin{aligned} &(1-x)^3 + 2016(1-x) = (y-1)^3 + 2016(y-1) \\ &\qquad \Rightarrow (1-x)^3 - (y-1)^3 = 2016\big[(y-1) - (1-x)\big]. \end{aligned}$$

Factorization gives

$$(2 - x - y)\big[(1-x)^2 + (1-x)(y-1) + (y-1)^2\big] = 2016(y + x - 2).$$

The above identity is true if and only if $y + x = 2$ since $(1-x)^2 + (1-x)(y-1) + (y-1)^2 > 0$. The correct answer is B.

35. Suppose $\log(a+b) = \log a + \log b$, for some real numbers a and b with $a > 1$ and $b > 1$. Find the value of $\log(a-1) + \log(b-1)$.

$A)$ 2 $B)$ 1 $C)$ 0 $D)$ 3 $E)$ 5

Solution: From $\log(a+b) = \log a + \log b$, we have $a + b - ab = 0$ and

$$\log(a-1) + \log(b-1) = \log(ab - a - b + 1) = \log 1 = 0.$$

The correct answer is C.

36. *Suppose $\{a_1,\ a_2,\ a_3,\cdots\}$ is a geometric sequence of real numbers. The sum of the first n terms is denoted by S_n. If $S_{10}=10$ and $S_{30}=70$, calculate the value of $\dfrac{S_{40}}{50}$.

A) 1 *B*) 2 *C*) 3 *D*) 5 *E*) 4

Solution: Denote the common ratio of $\{a_n\}$ as q. We have

$$S_{10}=\frac{a_1(q^{10}-1)}{q-1}=10 \quad \text{and} \qquad S_{30}=\frac{a_1(q^{30}-1)}{q-1}=70.$$

Division gives $\dfrac{S_{30}}{S_{10}}=\dfrac{q^{30}-1}{q^{10}-1}=q^{20}+q^{10}+1=7$. This means $q^{20}+q^{10}-6=0$. Therefore $q^{10}=2$ which means $\dfrac{a_1}{q-1}=10$.

$$S_{40}=\frac{a_1(q^{40}-1)}{q-1}=10(2^4-1)=150$$

The correct answer is C.

37. Simplify the expression $\sqrt{x+1-2\sqrt{x}}+\sqrt{x+1+2\sqrt{x}}$ for $0\leqslant x\leqslant 1$.

A) x^2 *B*) $2\sqrt{x}$ *C*) x *D*) 2 *E*) 1

Solution: Obviously $x\geqslant 0$. Rewrite

$$f(x)=\sqrt{(\sqrt{x}-1)^2}+\sqrt{(\sqrt{x}+1)^2}=1-\sqrt{x}+\sqrt{x}+1=2.$$

The correct answer is D.

38. If $\tan(\theta)<0$ and $\sin(\theta)=-\frac{1}{3}$, find $\cos(\theta)$.

A) $\sqrt{3}$ *B*) $\dfrac{\sqrt{3}}{3}$ *C*) $-\dfrac{\sqrt{3}}{3}$ *D*) $\dfrac{2\sqrt{2}}{3}$ *E*) $-\dfrac{2\sqrt{2}}{3}$

Solution: Since $\sin(\theta)=-\frac{1}{3}<0$, angle θ is in the third quadrant or fourth quadrant. But noting that $\tan(\theta)<0$ implies θ is in fourth quadrant. Therefore $\cos(\theta)\geqslant 0$. By trigonometric identity $\sin^2(\theta)+\cos^2(\theta)=1$, we have $\cos(\theta)=\sqrt{1-\sin^2(\theta)}=\sqrt{\frac{8}{9}}$. The correct answer is D.

39. The function $f(x)=ax+b$ satisfies $f(f(f(1)))=29$ and $f(f(f(0)))=21$. Find the value of $a+b$.

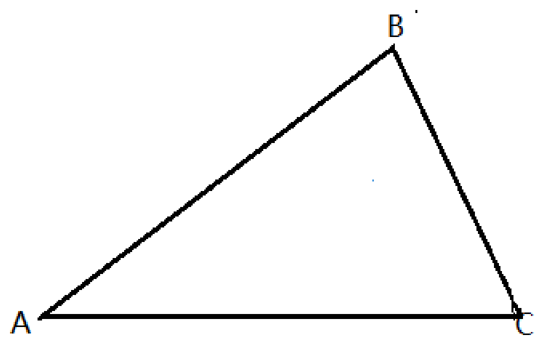
A) 0 *B*) 1 *C*) 2 *D*) 4 *E*) 5

Solution: Since

$$\begin{aligned}f(f(f(x)))&=a(a(ax+b)+b)+b=a^3x+a^2b+ab+b\\&\Rightarrow a^3=f(f(f(1)))-f(f(f(0)))=8\Rightarrow a=2\Rightarrow b=3,\end{aligned}$$

the correct answer is E.

40. *In triangle $\triangle ABC$, if $\angle A = 60°$, $AB = 2$, and $AC = 3$, find the length of the third side $\overline{BC}$.



$A)\ \sqrt{3}$ $\quad B)\ 2$ $\quad C)\ \sqrt{5}$ $\quad D)\ \sqrt{6}$ $\quad E)\ \sqrt{7}$

Solution: Using the Law of Cosines, we have $BC^2 = AB^2+AC^2-2AB\cdot AC\cos(60°) = 2^2+3^2-2(2)(3)\frac{1}{2} = 7$. The correct answer is E.

41. *If $\frac{5\pi}{4} \leqslant \theta \leqslant \frac{3\pi}{2}$, find the value of $\sqrt{1-\sin 2\theta} - \sqrt{1+\sin 2\theta}$.

$A)\ 2\sin\theta$ $\quad B)\ -2\sin\theta$ $\quad C)\ 2\cos\theta$ $\quad D)\ -2\cos\theta$ $\quad E)\ 0$

Solution: Rewrite

$$\begin{aligned}\sqrt{1-\sin 2\theta} - \sqrt{1+\sin 2\theta} &= |\cos\theta - \sin\theta| - |\cos\theta + \sin\theta| \\ &= \cos\theta - \sin\theta + \cos\theta + \sin\theta = 2\cos\theta.\end{aligned}$$

The correct answer is C.

42. *Find the minimum value of x^2+y^2 such that x and y satisfy the equation $(x+1)^2+(y-1)^2 = 8$.

$A)\ 2$ $\quad B)\ 1$ $\quad C)\ 0$ $\quad D)\ 3$ $\quad E)\ 4$

Solution: Let $x = -1+2\sqrt{2}\sin t$, $y = 1+2\sqrt{2}\cos t$. We have

$$\begin{aligned}x^2+y^2 &= (-1+2\sqrt{2}\sin t)^2 + (1+2\sqrt{2}\cos t)^2 \\ &= 1-4\sqrt{2}\sin t + 8\sin^2 t + 1 + 4\sqrt{2}\cos t + 8\cos^2 t = 10 - 8\sin\left(t-\frac{\pi}{4}\right).\end{aligned}$$

The correct answer is A.

43. *Find the range of $f(x) = 8(\sin^4 x + \cos^4 x - \sin x \cos x)$ for all real numbers x.

A) $(0, 9)$ *B*) $[0, 9)$ *C*) $(0, 9]$ *D*) $[0, 9]$ *E*) $[-8, 8]$

Solution: Using $\sin 2x = 2\sin x \cos x$, function $f(x)$ can be written as

$$f(x) = 8\left(\left(\sin^2 x + \cos^2 x\right)^2 - 2\sin^2 x \cos^2 x - \sin x \cos x\right) = 8 - 4\sin^2 2x - 4\sin 2x$$
$$= 9 - (2\sin 2x + 1)^2.$$

Hence, $\min_{0\leqslant x<\pi} f(x) = 0$ and $\max_{0\leqslant x<\pi} f(x) = 9$. The correct answer is *D*.

44. Find $\lim\limits_{x\to 0} \dfrac{\sqrt[3]{3x+8}-2}{x}$.

A) $\dfrac{5}{12}$ *B*) $\dfrac{7}{12}$ *C*) $\dfrac{1}{12}$ *D*) $\dfrac{1}{2}$ *E*) $\dfrac{1}{4}$

Solution: Let $u = \sqrt[3]{3x+8}$, we have

$$\frac{\sqrt[3]{3x+8}-2}{x} = \frac{u-2}{x} = \frac{(u-2)(u^2+2u+4)}{x(u^2+2u+4)} = \frac{u^3-8}{x(u^2+2u+4)} = \frac{3x}{x(u^2+2u+4)}$$
$$= \frac{3}{u^2+2u+4}.$$

Hence,

$$\lim_{x\to 0} \frac{\sqrt[3]{3x+8}-2}{x} = \lim_{u\to 2} \frac{3}{u^2+2u+4} = \frac{3}{12} = \frac{1}{4}.$$

The correct answer is *E*.

45. *The integers x and y satisfy the following system

$$\begin{cases} 5^x - \log_2(y+3) &= 3^y \\ 5^y - \log_2(x+3) &= 3^x \end{cases}$$

Find the value of $x + y$.

A) -2 *B*) -1 *C*) 0 *D*) 2 *E*) 3

Solution: From the system, we claim that $x > -3$ and $y > -3$. Subtracting the two equations and using the fact that function $5t + 3t + \log_2(t+3)$ is increasing on its domain $(-3, \infty)$, we can show that $x = y$. To solve $5^x - \log_2(x+3) = 3^x$, we note that $5^x > 4^x + 3^x$ and $4^x > \log_2(x+3)$ for $x > 2$. The only possible values of x are $-2, -1, 0$ or 1.

Direct substitution confirms that only $x = 1$ works. Hence, there is only one pair $(x, y) = (1, 1)$.

The correct answer is *D*.

46. *Let $f(x)$ be a real-valued function defined on $(-\infty, \infty)$ such that $f(f(x)) = 2^x - 1$. Find the value of $f(0) + f(1)$.

$A)$ 1 $B)$ 2 $C)$ 0 $D)$ -2 $E)$ -1

Solution: Note that

$$f(f(x)) = 2^x - 1 \Rightarrow f(f(f(x))) = f(2^x - 1) = 2^{f(x)} - 1$$
$$\Rightarrow f(0) = 2^{f(0)} - 1, f(1) = 2^{f(1)} - 1.$$

Consider the equation $y = 2^y - 1$, which has only solutions $y = 0$ and $y = 1$. Therefore $f(0) = 0$ and $f(1) = 1$ or $f(0) = 1$ and $f(1) = 0$. The correct answer is A.

47. *Find the number of ordered pairs of real numbers (a, b) such that $(a+bi)^{2016} = a - bi$, where $i = \sqrt{-1}$.

$A)$ 2014 $B)$ 2015 $C)$ 2016 $D)$ 2017 $E)$ 2018

Solution: Let $z = a + bi$, $\overline{z} = a - bi$. Then $(a + bi)^{2016} = a - bi$ implies that $z^{2016} = \overline{z}$.

$$|z|^{2016} = |z^{2016}| = |\overline{z}| = |z|.$$

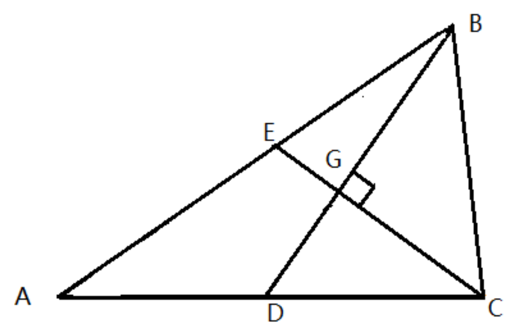
so

$$|z|(|z|^{2015} - 1) = 0.$$

Hence $|z| = 0$, and $(a, b) = (0, 0)$, or $|z| = 1$. In the case $|z| = 1$, using $z^{2016} = \overline{z}$, we have $z^{2017} = z\overline{z} = |z|^2 = 1$. Since $z^{2017} = 1$ has 2017 distinct solutions, there are $1 + 2017 = 2018$ ordered pairs that meet the required conditions.

The correct answer is E.

48. In the triangle $\triangle ABC$, $AB = 2\sqrt{13}$, $AC = \sqrt{73}$, E and D are the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. $\overline{BD}$ is perpendicular to $\overline{CE}$. Find the length of $\overline{BC}$.



$A)$ 3 $B)$ 4 $C)$ 5 $D)$ 6 $E)$ 7

Solution: Let G be the point of intersection of $\overline{BD}$ and $\overline{CE}$. Then $BC = \sqrt{BG^2 + CG^2}$, $BG = \frac{2}{3}BD$, and $CG = \frac{2}{3}CE$. We need to find the lengths of $\overline{BD}$ and $\overline{CE}$. Let $BG = x, CG = y$. Then $GD = \frac{1}{2}x$ and $GE = \frac{1}{2}y$. In right triangles $\triangle BGE$ and $\triangle CGD$, we have

$$\begin{aligned} x^2 + \frac{1}{4}y^2 &= 13, \\ \frac{1}{4}x^2 + y^2 &= \frac{73}{4}. \end{aligned}$$

Therefore $x = 3$ and $y = 4$. We have $BC = \sqrt{x^2 + y^2} = 5$. The correct answer is C.

49. If $0 \leqslant \theta \leqslant \pi$ and $x + \frac{1}{x} = 2\cos\theta$, find the value of $x^3 + \frac{1}{x^3}$.

$A)$ $2\cos\theta$ $\quad$ $B)$ $2\cos 3\theta$ $\quad$ $C)$ $3\cos 3\theta$ $\quad$ $D)$ $8\cos^3\theta$ $\quad$ $E)$ $3\cos^3\theta$

Solution: From $x + \frac{1}{x} = 2\cos\theta$, we have $x^3 + \frac{1}{x^3} + 3x^2\frac{1}{x} + 3x\frac{1}{x^2} = 8\cos^3\theta$ and then $x^3 + \frac{1}{x^3} = 8\cos^3\theta - 3\left(x + \frac{1}{x}\right) = 8\cos^3\theta - 6\cos\theta = 2(4\cos^3\theta - 3\cos\theta) = 2\cos 3\theta$. The correct answer is B.

50. *Find the number of triples (x, y, z) of rational numbers which are solutions of the following system.

$$\left\{ \begin{aligned} x + y + z &= 0 \\ xyz + z &= 0 \\ xy + yz + zx + y &= 0 \end{aligned} \right.$$

$A)$ 1 $\quad$ $B)$ 2 $\quad$ $C)$ 3 $\quad$ $D)$ 4 $\quad$ $E)$ 5

Solution: If $z = 0$, the original system reduces to

$$\left\{ \begin{aligned} x + y &= 0 \\ xy + y &= 0 \end{aligned} \right.$$

We have

$$\left\{ \begin{aligned} x &= 0 \\ y &= 0 \end{aligned} \right. \quad \text{or} \quad \left\{ \begin{aligned} x &= -1 \\ y &= 1 \end{aligned} \right.$$

If $z \neq 0$, from $xyz + z = 0$, we have $xy = -1$. From $x + y + z = 0$, we obtain $z = -x - y = \frac{1}{y} - y$. Substituting this into the third equation, we have

$$0 = xy + yz + zx + y = -1 + y\left(\frac{1}{y} - y\right) - \frac{1}{y}\left(\frac{1}{y} - y\right) + y = \frac{1}{y^2}(y-1)\left(y^3 - y - 1\right).$$

Therefore

$$\begin{cases} y = 1 \\ x = -1 \\ z = 0 \end{cases}$$

which is contrary to the condition $z \neq 0$. The equation $y^3 - y - 1 = 0$ has no rational solutions. The total number of rational solutions is 2. The correct answer is B.