Forty First Annual Columbus State University
Invitational Mathematics Tournament

Sponsored by
The Columbus State University
Department of Mathematics
March 7, 2015
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The Columbus State University Mathematics faculty welcomes you to this year's tournament and to our campus. We wish you success on this test and in your future studies.

Instructions

This is a 90-minute, 50-problem, multiple-choice exam. There are five possible responses to each question. You should select the one "best" answer for each problem. In some instances this may be the closest approximation rather than an exact answer. You may mark on the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer circle the choice you have made on the test booklet. Carefully transfer your answers to the score sheet. Completely darken the blank corresponding to the letter of your response to each question. Mark your answer boldly with a No. 2 pencil. If you must change an answer, completely erase the previous choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response. The examination will be scored on the basis of +12 for each correct answer, −3 for each incorrect selection, and 0 for each omitted item. Each student will be given an initial score of +200.

Pre-selected problems will be used as tie-breakers for individual awards. These problems, designated with an asterisk (*), in order of consideration are: 5, 10, 11, 12, 14, 17, 20, 21, 23, 26, 27, 28, 30, 31, 32, 35, 37, 41, 47, and 48.

Throughout the exam, $\overline{AB}$ will denote the line segment from point A to point B and AB will denote the length of $\overline{AB}$. Pre-drawn geometric figures are not necessarily drawn to scale. The measure of the angle $\angle ABC$ is denoted by $m\angle ABC$.

Review and check your score sheet carefully. Your student identification number and your school number must be encoded correctly on your score sheet.

When you complete your test, bring your pencil, scratch paper and answer sheet to the test monitor. Leave the room after you have handed in your answer sheet. Please leave quietly so as not to disturb the other contestants. Do not congregate outside the doors by the testing area. You may keep your copy of the test. Your sponsor will have a copy of solutions to the test problems.

Do not open your test until instructed to do so!

1) What is the 2015th digit (to the right of decimal point) in the decimal expansion of $\frac{224}{1111}$?

A) 0 $\boxed{\text{B) 1}}$ C) 2 D) 5 E) 6

Solution: $\frac{224}{1111} = 0.\overline{2016}$; since $2015 = 4\times 503 + 3$ we conclude that the 2015th digit of $\frac{224}{1111}$ is 1.

2) If the three zeros of the polynomial $p(x) = x^3 + bx^2 + 623x - 2015$ are positive integers, what is the value of b?

A) 33 B) -33 $\boxed{\text{C) } -49}$ D) 49 E) 142

Solution: Notice that $2015 = 5\times 13\times 31$; that is, $p(x) = (x-5)(x-13)(x-31)$ and $b = -49$

3) What is the largest number of points in which the graphs of a third degree polynomial and a fifth degree polynomial can meet?

A) 3 $\boxed{\text{B) 5}}$ C) 8 D) 2 E) 15

Solution: The difference between the polynomials is a polynomial of degree five and its number of real zeroes is at most five.

4) Which of the following numbers is not a divisor of 2015^5-2015?

A) 16 B) 32 C) 256 D) 512 $\boxed{\text{E) Both, C) and D)}}$

Solution: Notice that $2015^5 - 2015 = 2015(2015-1)(2015+1)\left(2015^2+1\right)$ or

$2015^5 - 2015 = 2014\times 2015\times 2016\times 4060226$

$2015^5 - 2015 = 2\times 19\times 53\times 5\times 13\times 31\times 2^5\times 3^2\times 7\times 2\times 2030113$

Which means, the largest power of 2 dividing 2015^5-2015 is $2^7 = 128$.

5) What is the number of ordered pairs (x, y) of positive integers that satisfy the equation $31x + 13y = 2015$?

A) 0 $\boxed{\text{B) 4}}$ C) 5 D) 6 E) Infinitely many pairs.

Solution: Notice that $y=155-\frac{31}{13}x$; since both x and y must be positive integers we need $x=13k$ with $k=1,2,\ldots$ and $155-\frac{31}{13}x>0$ which means that the largest value of k must be 4. We conclude that there four such pairs.

6) Find the sum of the solutions of the equation $\sqrt{2x+7}+4=x$.

$\boxed{\text{A) 9}}$ B) -9 C) 10 D) -10 E) None of these

Solution: $\sqrt{2x+7}+4=x$ is equivalent to $\sqrt{2x+7}=x-4$. Square each side to obtain the equation (no longer equivalent to the original one) $2x+7=\left(x-4\right)^2$; this one is equivalent to $x^2-10x+9=(x-9)(x-1)=0$. Verifying using the original equation, we see that only $x=9$ is a solution, therefore the sum is 9.

7) At the end of the 1998 season, the National Football League's all-time leading passer during regular season play was Dan Marino with 4763 completed passes out of 7989 attempts. In his debut 1998 season, Peyton Manning made 326 completed passes out of 575 attempts. What is the smallest number of consecutive completed passes that Peyton Manning would have to make to exceed Dan Marino's pass completion percentage?

A) 41.7 $\boxed{\text{B) 42}}$ C) 42.5 D) 43 E) None of these

Solution: Let x denote the number of consecutive passes needed; then we require that $\frac{326+x}{575+x}>\frac{4763}{7989}$; solving for x we get that $x>41.634$ which means that the smallest number (integer) that will make $\frac{326+x}{575+x}>\frac{4763}{7989}$ is $x=42$.

8) On Tuesday the price of gas was 5% more expensive that on Monday. On Wednesday the price went down by 2%. If a gallon of gas cost \$ 3.59 on Wednesday, how much did it cost on Monday? Please, round your answer to two decimal places.

A) \$3.35 B) \$2.44 C) \$3.40

D) \$3.48 $\boxed{\text{E) \$ 3.49}}$

Solution: Let x be the price of gas on Monday. On Tuesday the new price is $1.05x$ and on Wednesday was 98% of Tuesday's price, that is, $0.98(1.05x)$. Thus $0.98(1.05x) = 3.50$, this means that on Monday gas cost \$ 3.49, rounded to two decimal places.

9) Let $\Re = \{x \in \mathbb{R} : x \neq -1\}$ and for all $x, y \in \Re$ define $x \square y = x + y + xy$. Find all solutions in $\Re$ of the equation $x \square x = 2 \square x$.

A) $x = 0$ B) $x = 0$ and $x = 2$ C) $x = -1$ and $x = 2$

$\boxed{\text{D) } x = 2}$ E) $x = 0$, $x = -1$ and $x = 2$

Solution: The equation $x \square x = 2 \square x$ means $x + x + x^2 = 2 + x + 2x$ or, equivalently, $(x-2)(x+1) = 0$. The possible values of x are $x = -1$ and $x = 2$. By definition, the only solution in $\Re$ is $x = 2$.

10) How many integers between 1 and 2015 are multiples of 3 or 4?

A) 1008 B) 1174 C) 1006

$\boxed{\text{D) 1007}}$ E) 1176

Solution: The inequality $3k \leq 2015$ is used to count the number of positive integers divisible by 3 between 1 and 2015. That is, $k \leq 2015/3 = 671.67$ tells us that there are 671 numbers with the desired property. Similarly, $4k \leq 2015$, implies that there are 503 numbers multiple of 4 between 1 and 2015.

Notice that the numbers between 1 and 2015 that are divisible by 12 are counted twice. Since $k \leq 2015/12 = 167.92$, we see that there are 167 such numbers. Finally, the number of integers between 1 and 2015 are multiples of 3 or 4 is 671+503 – 167 = 1007.

11) An island has three kinds of inhabitants, knights, who always tell the truth, knaves, who always lie, and spies who can either lie or tell the truth. You encounter three people A, B, and C. You know one of these people is a knight, one is a knave, and one is a spy. A says "C is the knave", B says "A is the knight", and C says "I am the spy". Which of the following is true?

$\boxed{\text{A) A is the knight}}$ B) B is the knight C) C is the knight

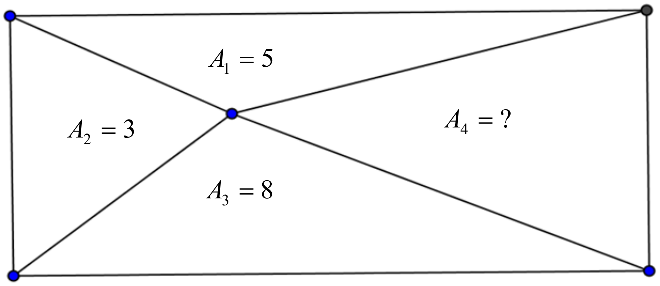
D) There is more than one solution E) There is no solution

Solution: The correct answer is A) A is the knight. Knights always tell the truth. B cannot be the knight since if B was the knight then B's statement "A is the knight" would be a lie (there is only one knight). C cannot be the knight since if C was the knight then C's statement "I am a spy" would be a lie. So A has to be the knight, furthermore A's statement "C is a knave" is the truth. C's statement "I am the spy" is a lie as it should be. B is the spy and decides to tell the truth "A is the knight".

12) In the rectangle shown in the figure, the area of the three the triangles are $A_1 = 5$, $A_2 = 3$, and $A_3 = 8$. Find the area A_4 of the remaining triangle.

A) 7 B) 8 C) 9

$\boxed{\text{D) 10}}$ D) 11



Solution: Let *a* and *b,* be the base and height, respectively, of the rectangle. Let h_1, h_3 be the heights of the corresponding triangles with areas A_1, A_3 with respect to the base *a*. Similarly, let h_2, h_4 be the heights of the corresponding triangles with areas A_2, A_4 with respect to the base *b*.

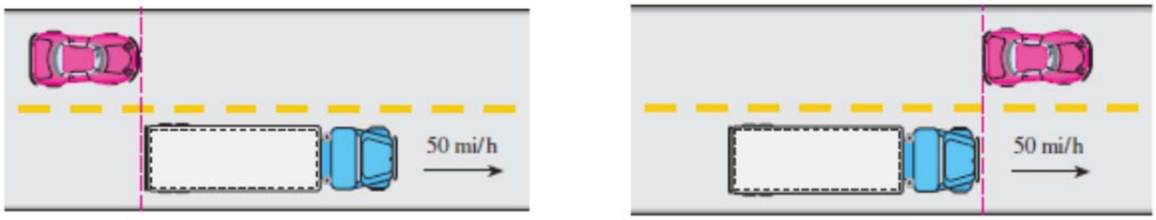
Since $A_1 + A_3 = \frac{1}{2}ah_1 + \frac{1}{2}ah_3 = \frac{1}{2}ab$ and $A_2 + A_4 = \frac{1}{2}bh_2 + \frac{1}{2}bh_4 = \frac{1}{2}ab$, we have $A_1 + A_3 = A_2 + A_4$ or $A_4 = 10$.

13) Find the remainder of $1006^{2015} + 1007^{2015} + 2016$ when divided by 2013.

A) 2014 B) 2015 C) 2 $\boxed{\text{D) 3}}$ E) 4

Solution: Using the fact that $1006^{2015} + 1007^{2015}$ is divisible by $1006 + 1007 = 2013$ we can easily see that the remainder is 3.

14) A woman driving a car 14 feet long is passing a truck 30 feet long. The truck is traveling at 50 mi/h. How fast must the woman drive her car so that she can pass the truck completely in 6 seconds, from the position shown in figure (a) to the position shown in figure (b)? Please, round your answer to two decimal places.



(a) (b)

A) 69.1 mi/h

B) 50.68 mi/h

C) 51 mi/h

D) 59.55 mi/h

$\boxed{\text{E) 51.59 mi/h}}$

Solution: Let v be the velocity of the car; set the origin for distances at the point in figure (a) when the front end of the car meets the rear end of the truck. If s_1 is the distance traveled by the car and s_2 is the distance traveled by the truck, then $s_1 = vt$ and $s_2 = 50t$, where t is in hours. In $6\sec = \frac{6}{3600} hour$ the rear end of the car must be at the front end of the truck as in figure (b). Using the fact that $14\, feet = \frac{14}{5280} miles$, we have $v \times \frac{6}{3600} = 50 \times \frac{6}{3600} + \frac{14}{5280}$ and therefore, $v = 51.59$ mi/h.

15) Find the center (h,k) of the circle on the xy-plane, passing through the points $(0,b)$, $(-a,0)$ and $(a,0)$.

A) $(h,k) = \left(0, \frac{a^2 - b^2}{2b}\right)$

B) $(h,k) = \left(0, \frac{b^2 + a^2}{2b}\right)$

C) $(h,k) = \left(0, \frac{b^2 - a^2}{b}\right)$

D) $(h,k) = \left(\frac{a^2 - b^2}{2b}, 0\right)$

$\boxed{\text{E) } (h,k) = \left(0, \frac{b^2 - a^2}{2b}\right)}$

Solution: By symmetry, the value of h must be 0. On the other hand we must have $r^2 = (b-k)^2$ and $r^2 = a^2 + k^2$, where r is the radius on the circle going through the three given points. A simple algebra shows that $k = \frac{b^2 - a^2}{2b}$.

16) A driver sets out on a journey. For the first half of the distance she drives at the leisurely pace of 30 mi/h; during the second half she drives 60 mi/h. What is her average speed on this trip?

A) 35 mi/h $\boxed{\text{B) 40 mi/h}}$ C) 45 mi/h

D) 50 mi/h E) 55 mi/h

Solution: Let d be the distance traveled on each half of the trip. Let t_1 and t_2 be the times taken for the first and second halves of the trip. For the first half of the trip we have $30 = \frac{d}{t_1}$ and for the second half, $60 = \frac{d}{t_2}$; if the average speed for the entire trip is v, then $v = \frac{2d}{t_1 + t_2}$ which can be written as

$$v = \frac{2d}{\frac{d}{30} + \frac{d}{60}} = 40.$$

17) It takes a paddle boat 50 minutes to travel 4 miles up a river and 4 miles back, going at a steady speed of 10 miles per hour (with respect to the water). Find the speed of the current in miles per hour.

$\boxed{\text{A) 2 mph}}$ B) 9.92 mph C) 2.90 mph

D) 3 mph E) None of these

Solution: Let v be the speed of the current, t_1 the time it takes to go up the river and t_2 the time to return. We have $t_1 + t_2 = 50$ minutes, with $t_1 = \frac{4}{10 - v}$ and $t_2 = \frac{4}{10 + v}$. Thus, using hours for the time unit, we have $\frac{50}{60} = \frac{4}{10 - v} + \frac{4}{10 + v}$; solving for v we get $v = 2$ mph.

18) Let $a > b > 0$, $a^2 + b^2 = 3ab$. Find the value of $\frac{a + b}{a - b}$.

A) $\sqrt{2}$ B) $\sqrt{3}$ $\boxed{\text{C) } \sqrt{5}}$ D) 2 E) 3

Solution: From $a > b > 0, a^2 + b^2 = 3ab$, we get $\left(\frac{a}{b}\right)^2 - 3\left(\frac{a}{b}\right) + 1 = 0$. Solving the equation for $\frac{a}{b}$, we have $\frac{a}{b} = \frac{3+\sqrt{5}}{2}$. Therefore, $\frac{a+b}{a-b} = \frac{\frac{a}{b}+1}{\frac{a}{b}-1} = \sqrt{5}$.

19) If $\sqrt{17-12\sqrt{2}} = a + b\sqrt{2}$, which of the following is $\left(a+b\sqrt{2}\right)^{-1}$?

A) $17+2\sqrt{2}$

B) $\frac{1}{17-2\sqrt{2}}$

C) $3-2\sqrt{2}$

$\boxed{\text{D) } 3+2\sqrt{2}}$

E) $2-3\sqrt{2}$

Solution: $\sqrt{17-12\sqrt{2}} = a + b\sqrt{2}$ implies $17-12\sqrt{2} = a^2 + 2ab\sqrt{2} + 2b^2$; this means that $a = 3$ and $b = -2$. Thus $\left(3-2\sqrt{2}\right)^{-1} = \frac{1}{\left(3-2\sqrt{2}\right)} \frac{\left(3+2\sqrt{2}\right)}{\left(3+2\sqrt{2}\right)} = \frac{\left(3+2\sqrt{2}\right)}{9-8} = 3+2\sqrt{2}$

20) Find all real solutions of the inequality $x^2 - 3 \le \left|3 - x^2\right|$

A) $-\sqrt{3} < x < \sqrt{3}$

B) $-\sqrt{3} \le x \le \sqrt{3}$

$\boxed{\text{C) } -\infty < x < \infty}$

D) $-3 < x < 3$

E) $-3 \le x \le 3$

Solution: $x^2 - 3 \le \left|3 - x^2\right|$ is equivalent to $x^2 - 3 \le 3 - x^2$ or $3 - x^2 \le -\left(x^2 - 3\right)$. The first inequality means $2\left(x^2 - 3\right) \le 0$ or that $-\sqrt{3} \le x \le \sqrt{3}$. The second inequality is equivalent to $0 \le 0$, which is true for all x. The solution set of $x^2 - 3 \le \left|3 - x^2\right|$ is the union of the two solution sets, namely, $-\infty < x < \infty$.

21) If the function $f(x) = \frac{1}{x^2 + 2x + c}$ is defined for all real numbers, then find all possible values of c.

$\boxed{\text{A) } c > 1}$

B) $c = 1$

C) $c < 1$

D) $c \le 1$

E) $c > 2$

Solution: Note that we need $x^2 + 2x + c \ne 0$ for any real number. Hence $x^2 + 2x + c = 0$ has no real zeros. Equivalently, $b^2 - 4ac = 4 - 4c < 0$, which indicates $c > 1$. Or $x^2 + 2x + c = (x+1)^2 + c - 1$ which means that we need $c > 1$.

22) Which of the following is the equation of the circle that goes through the origin and is tangent to the line $-x+y=8$ at the point $(0,8)$?

A) $(x+4)^2+(y+4)^2=\sqrt{32}$ B) $(x+4)^2+(y+4)^2=32$ C) $(x-4)^2+(y-4)^2=16$

D) $(x-4)^2+(y-4)^2=\sqrt{32}$ $\boxed{\text{E) } (x-4)^2+(y-4)^2=32}$

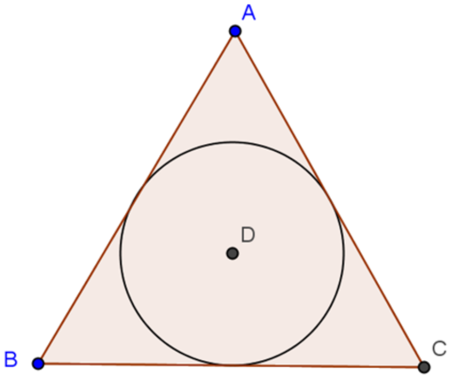
Solution: Since the circle is tangent to the given line at $(0,8)$, we then know that its center must be on the line perpendicular to $-x+y=8$ at $(0,8)$; that is, the center, say, (h,k) is on the line $x+y=8$. If we let r be the radius of the circle, then we must have $r^2=h^2+k^2$, because the origin is on the circle, and $r^2=(h-0)^2+(k-8)^2=h^2+64-16k+k^2$. Therefore, $k=4$ and consequently, $h=4$. Thus, the center of the circle is at $(4,4)$ and the equation of the circle is $(x-4)^2+(y-4)^2=32$

23) Suppose that $r_1,r_2,\ldots,r_{2015}$ are the 2015 roots of the polynomial $p(x)=x^{2015}+2014x+2016$. Find the average of $r_1^{2015},r_2^{2015},\ldots,r_{2015}^{2015}$.

A) -2014 B) 2014 C) 2016 $\boxed{\text{D) -2016}}$ E) 2015

Solution: Since $p(r_i)=r_i^{2015}+2014r_i+2016=0$ for all $i=1,2,\ldots,2015$ we have $\sum_{i=1}^{2015} r_i^{2015}+2014\sum_{i=1}^{2015} r_i+2015\times 2016=0$; we also see that $\sum_{i=1}^{2015} r_i=0$ because the coefficient of x^{2014} is zero in the given polynomial. Hence, $\frac{1}{2015}\sum_{i=1}^{2015} r_i^{2015}=-2016$

24) The area of the equilateral triangle shown in the figure on the right is 6. Find the area of the inscribed circle.



$\boxed{\text{A) } 2\pi/\sqrt{3}}$ B) 2π C) $\sqrt{3}\pi$

D) $\pi/\sqrt{3}$ E) $\sqrt{3}\pi/2$

Solution: If we let a be the length of the side of the triangle, using Pythagoras' theorem to get the height of the triangle, we get that its area is given by $A_T = \frac{\sqrt{3}}{4}a^2$; on the other hand, the area of the circle in terms of a is given by $A_C = \frac{\pi}{12}a^2$. Since $A_T = 6 = \frac{\sqrt{3}}{4}a^2$, we get $A_C = \frac{2}{\sqrt{3}}\pi$.

25) What is the smallest value of $x + y$ if x and y are positive integers such that $\frac{1}{x} + \frac{1}{2x} + \frac{1}{3x} = \frac{1}{y^2 - 2y}$?

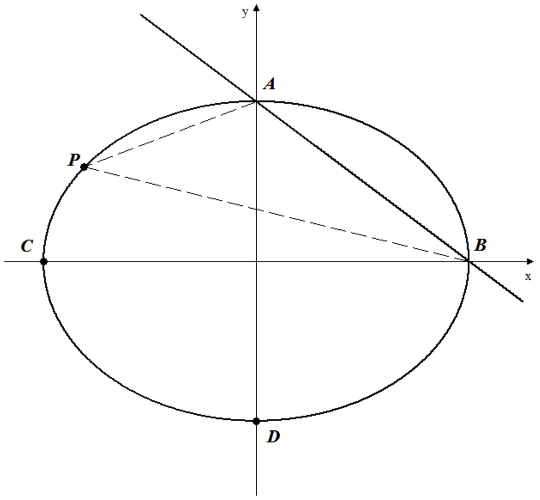
A) 6 B) 44 $\boxed{\text{C) 50}}$ D) 56 E) 82

Solution: It is easy to see that the restriction on x and y is equivalent to $11(y^2 - 2y) = 6x$. Since 11 must divide x, we have $11(y^2 - 2y) = 6k11$ or $(y^2 - 2y) = 6k$ for some positive integer k. This means 6 divides $y^2 - 2y$. The smallest positive integer y for which 6 divides $y^2 - 2y$ is $y = 6$. Thus, with $y^2 - 2y = 24 = 6k$ we get $k = 4$ which means $x = 11k = 44$. The minimum value of $x + y$ is 50.

26) The straight line $\frac{x}{4} + \frac{y}{3} = 1$ and the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ intersect at two points *A* and *B*. A third point *P* is selected on the ellipse in such way that $\triangle PAB$ has area 4. How many points *P* on the ellipse can be selected in the same way?

A) 1 $\boxed{\text{B) 2}}$ C) 3 D) 4 E) Infinitely many points

Solution: It is easy to see that the two intersection points are $A = (0,3)$ and $B = (4,0)$. A picture of the situations is as follows:



If we let h be the height of the triangle $\triangle PAB$ with respect to the base $\overline{AB}$, then $4 = \frac{1}{2}AB \times h = \frac{1}{2}5 \times h$; this means that $h = 1.6$. Now observe that when *P* slides on the ellipse to become the point *A*, the area of $\triangle PAB$ is 0 ($h = 0$) and when *P* becomes *C*, the area of $\triangle PAB$ is 12 ($h = 4.8$); then there is only one point *P* on the arc $\overset{\frown}{CA}$ of the ellipse that will define a triangle $\triangle PAB$ with area 4. The same argument applies to a point *P* on the arc $\overset{\frown}{DB}$ of the ellipse. For *P* on the arc between *C* and *D*, we $h > 1.6$ and for *P* on the arc between *A* and *B*, we have $h < 1.6$. This is because the area of a quarter of the ellipse is 3π and the area of the triangle formed by A, B and the origin, is 6 and $3\pi-6\approx 2.15<4$. Thus, there are only two points with the desire property.

27) There are five different products with the same price. Beginning today, every day each product will get either a 10% or 20% discount. Let r be the ratio of the highest price to the lowest price. On the days in which the five prices are all different, what is the minimum value of r?

A) $\left(\frac{9}{8}\right)^3$ $\boxed{\text{B) } \left(\frac{9}{8}\right)^4}$ C) $\left(\frac{9}{8}\right)^5$ D) $\frac{9}{8}$ E) $\left(\frac{9}{8}\right)^2$

<u>Solution:</u> Let a be the price before discount. After n days, the price for each product can be expressed as $a(1-10\%)^k(1-20\%)^{n-k}=a\left(\frac{9}{10}\right)^k\left(\frac{8}{10}\right)^{n-k}$ for $0\le k\le n$. For r to be smallest, the prices for the five products should be $a\left(\frac{9}{10}\right)^k\left(\frac{8}{10}\right)^{n-k}$, $a\left(\frac{9}{10}\right)^{k+1}\left(\frac{8}{10}\right)^{n-k-1}$, $a\left(\frac{9}{10}\right)^{k+2}\left(\frac{8}{10}\right)^{n-k-2}$, $a\left(\frac{9}{10}\right)^{k+3}\left(\frac{8}{10}\right)^{n-k-3}$ and $a\left(\frac{9}{10}\right)^{k+4}\left(\frac{8}{10}\right)^{n-k-4}$, respectively, for some k with $0\le k\le n$. Then

$$r=\frac{a\left(\frac{9}{10}\right)^{k+4}\left(\frac{8}{10}\right)^{n-k-4}}{a\left(\frac{9}{10}\right)^{k}\left(\frac{8}{10}\right)^{n-k}}=\left(\frac{9}{8}\right)^4.$$

28) Let x, y be positive integers so that $\sqrt{x-7}+\sqrt{x+2}=y$. What is the value of y?

$\boxed{\text{A) } 9}$ B) 3 C) 1 D) 5 E) 6

<u>Solution:</u> Since x, y be positive integers, $n=\sqrt{x-7}$ and $m=\sqrt{x+2}$ are positive integers with $m>n$. Hence $\begin{cases} m^2=x+2 \\ n^2=x-7 \end{cases}$. We have $(m-n)(m+n)=9$. Note that $m-n$ and $m+n$ have the same parity, i.e., either both of them are even or both of them are odd. Therefore, there is only one possibility, $\begin{cases} m+n=9 \\ m-n=1 \end{cases}$, which gives 9 as the answer.

29) There are two positive solutions to the equation $\log_{2x} 2 + \log_4 2x = -\frac{3}{2}$. What is the product of the two solutions?

A) $1/21$ B) $3/21$ C) $\boxed{1/32}$ D) $1/8$ E) 2

Solution: By changing the bases in the log functions we get $\log_{2x} 2 = \frac{\log_2 2}{\log_2 2x} = \frac{1}{1+\log_2 x}$ and $\log_4 2x = \frac{\log_2 2x}{\log_2 4} = \frac{1+\log_2 x}{2}$. Thus, the original equation is the same as $\frac{1}{1+\log_2 x} + \frac{1+\log_2 x}{2} = -\frac{3}{2}$. Solving for $\log_2 x$ we get that $\log_2 x = -3$ or $\log_2 x = -2$, meaning that $x = \frac{1}{8}$ or $x = \frac{1}{4}$. The product of the solutions is $1/32$.

30) Let $a > 0$, $a \neq 1$, and $\frac{p}{q} \neq 0$ a rational number. Which of the following functions is equal to the function $f(x) = \log_{a^{p/q}} x$, $x > 0$?

A) $\log_a x^{p/q}$ B) $-\frac{p}{q}\log_a x$ C) $\log_a x^{-q/p}$

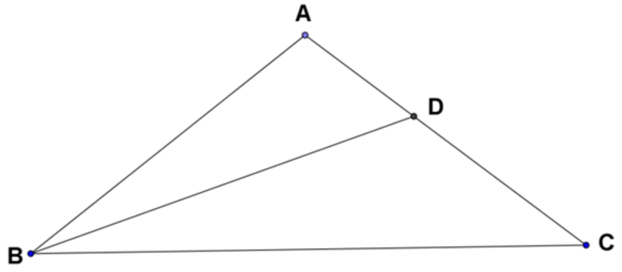
$\boxed{\text{D) } \log_a x^{q/p}}$ E) $\frac{p}{q}\log_a x$

Solution: By changing the bases we get $\log_{a^{p/q}} x = \frac{\log_a x}{\log_a a^{p/q}} = \frac{\log_a x}{\frac{p}{q}\log_a a} = \frac{q\log_a x}{p} = \log_a x^{q/p}$.

31) The isosceles triangle ΔABC shown in the figure has the property that $\angle BAC = 100^{\circ}$, $AB = AC$ and BD is the bisector of the angle $\angle ABC$. If $BD = 7.5$ and $AD = 2.5$, find BC.

A) $\boxed{10}$ B) 9.5 C) 18.75

D) 11.5 E) None of these



Solution:

Find E and F on $\overline{BC}$, such that $BE = AB$ and $BF = BD$.

Since $\angle A = 100^o$, we have $\angle ABC = \angle C = 40^o$.

Also, $\angle ABD = \angle DBC = 20^o$. implies $\angle BFD = 80^o$

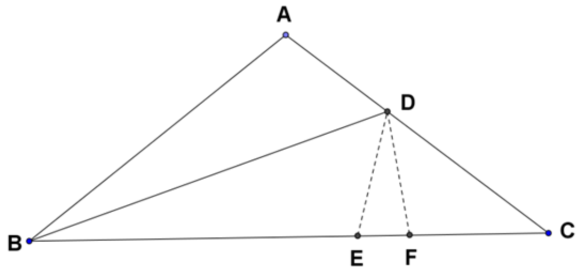
$\angle C = 40^o$ implies $\angle FDC = 40^o$, so $DF = FC$

$\Delta ABD \cong \Delta EBD$ implies $\angle DEB = \angle A = 100^o$,

so $\angle DEF = 80^o$.

$\therefore \quad DE = DF = AD$

and $BC = BF + FC = BD + AD = 10$



32) It is claimed that 40% of Americans do not have any health insurance. In a randomly selected group of three people, what is the probability that only one of them has health insurance?

A) 0.096 [✓ B) 0.288] C) 0.712

D) 0.144 E) 0.432

Solution: Obviously, the order of the selection is not important. If we let p be the probability that the selected person has health insurance and q the probability that does not, then the answer is given by $pqq + qpq + qqp = 3 \times 0.6 \times 0.4^2 = 0.288$.

33) Sue owns 5 different skirts, 7 different blouses, and 6 different pairs of slacks. The only combination she will not wear is her orange blouse with her pink skirt or her blue slacks. Beginning on the first day of school, she decides to wear a different outfit on each school day as long as possible. On what day must she finally wear an outfit that she has already worn?

A) 77th B) [✓ 76th] C) 74th D) 75th E) None of these

Solution: The number of outfits with skirts and blouses is $4 \times 7 + 1 \times 6 = 34$ and the number of outfits with blouses and slacks is $6 \times 6 + 1 \times 5 = 41$. This gives a total of 75 outfits for 75 different days. On the 76^{th} day she will repeat an outfit.

34) 12 points lie in a plane in such a way that exactly 5 of the points are on one straight line and apart from these 5 points, no three points lie on a straight line. Find the total number of distinct triangles that can be drawn with vertices on the 12 points.

A) 105 B) 175 [✓ C) 210] D) 455 E) None of these

Solution: The number of triangles with vertices off the straight line containing the five points is $\binom{7}{3} = \frac{7!}{3!4!} = 35$. The number of triangles with two vertices off the line and one on the line is $\binom{7}{2}\binom{5}{1} = \frac{7!}{2!5!}5 = 105$ and the number of triangles with one vertex off the line and two on the line is

$\binom{7}{1}\binom{5}{2} = 7\frac{5!}{2!3!} = 70$. Since these options cover all possible choices of vertices, we have that the total number of triangles is 35+105+70 = 210.

35) Let S_1 denote the area of the equilateral triangle $\Delta A_1B_1C_1$ and S_2 the area of the equilateral triangle $\Delta A_2B_2C_2$ whose vertices are the centers of the three largest circles of the same radius tangent to each other and the sides of $\Delta A_1B_1C_1$ as shown in the figure below. Find $\frac{S_2}{S_1}$.

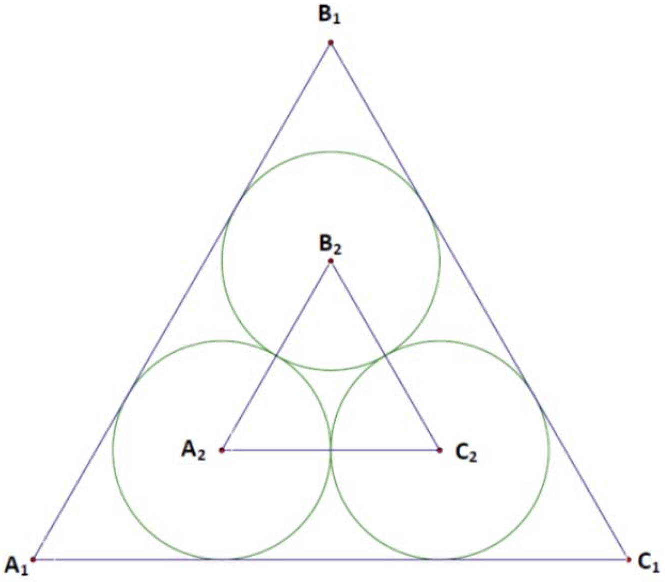
A) $\frac{1}{1+\sqrt{3}}$

B) $\frac{1}{2\left(1+\sqrt{3}\right)}$

C) $\frac{1}{2\left(1+\sqrt{3}\right)^2}$

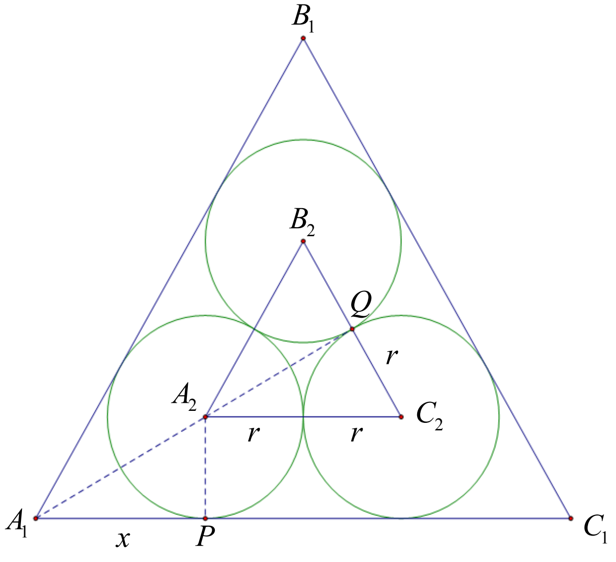
$\boxed{\text{D) } \frac{1}{\left(1+\sqrt{3}\right)^2}}$

E) $\frac{2}{\left(1+\sqrt{3}\right)^2}$



Solution: From the figure on the right we see that $\frac{A_2P}{A_1A_2} = \frac{QC_2}{A_2C_2}$ which gives us $A_1A_2 = 2r$ and therefore, by Pythagoras, $x = \sqrt{3}r$. If we let $a_i = A_iC_i$ for $i = 1, 2$, then we have that $a_1 = \sqrt{3}r + 2r + \sqrt{3}r = 2(1+\sqrt{3})r$ and $a_2 = 2r$. This shows that $\frac{a_2}{a_1} = \frac{1}{\left(1+\sqrt{3}\right)}$.

Since the areas of the triangles are proportional to the squares of these sides, we have $\frac{S_2}{S_1} = \frac{1}{\left(1+\sqrt{3}\right)^2}$.



36) Consider the equilateral triangle $\Delta A_1B_1C_1$ shown on the picture below. Let A be the intersection of the angle bisectors. Construct the equilateral triangle $\Delta A_2B_2C_2$ whose vertices A_2, B_2, C_2 are the midpoints of the segments $\overline{AA_2}, \overline{AB_2}, \overline{AC_2}$, respectively. Construct now, inside the second triangle, a third triangle $\Delta A_3B_3C_3$ using the same method. Repeat this process 2015 times to get a sequence of triangles $\Delta A_iB_iC_i \quad i = 1, 2, 3, \ldots, 2015$. If the first triangle $\Delta A_1B_1C_1$ has perimeter 3, what is the sum of the perimeters of all the triangles in the sequence?

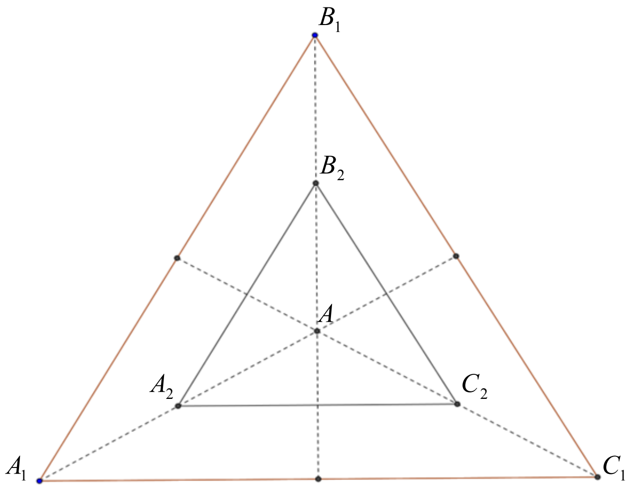
A) $\frac{6(2^{2016}-1)}{2^{2015}}$

B) $\frac{3(2^{2015}-1)}{2^{2014}}$

C) $\boxed{\frac{3(2^{2016}-1)}{2^{2015}}}$

D) $\frac{6(2^{2015}-1)}{2^{2014}}$

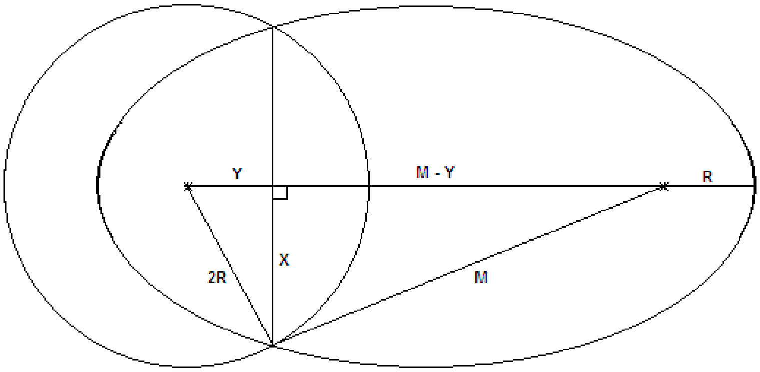
E) $\frac{3(2^{2015}-1)}{2^{2015}}$



Solution: From the basic geometry of the equilateral triangle we can see that the perimeter of a triangle in the sequence is half the perimeter of the previous triangle. Thus, the sum of all the perimeters is given by $3\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{2015}}\right)=\frac{3(2^{2016}-1)}{2^{2015}}$.

37) Suppose that the distance between the foci determining a given ellipse is M, and the total distance from any point on the ellipse to the two foci is $M + 2R$. Suppose further that we have constructed, as shown in the figure below, a circle of radius $2R$ and having its center located at one of the two foci of the ellipse. Find the length L of the line segment that connects the two points of intersection of the ellipse and circle.

$\boxed{\text{A)}\ L=\frac{4R\sqrt{M^2-R^2}}{M}}$



B) $L=\frac{R\sqrt{M^2-R^2}}{M}$

C) $L=\frac{2R\sqrt{M^2-R^2}}{M}$

D) $L=\frac{4R\sqrt{M^2+R^2}}{M}$

E) $L=\frac{2\sqrt{M^2+R^2}}{M}$

Solution: Using the Pythagorean rule, we get $X^2+Y^2=(2R)^2$, and $X^2+(M-Y)^2=M^2$. Substituting into the second equation, we get $(2R)^2 - Y^2+(M-Y)^2=M^2$. Solving this equation, we have

$4R^2 - Y^2 + M^2 - 2MY + Y^2 = M^2$ from which we get that $Y = 2R^2 / M$.

Substituting back into the equation $X^2 + (2R^2 / M)^2 = (2R)^2$, we get $X^2 = (2R)^2 - (2R^2 / M)^2$ and finally, that $X = [2R\ \surd(M^2- R^2)\,]/ M$ and $L = 2X = [4R\ \surd(M^2- R^2)\,]/ M$

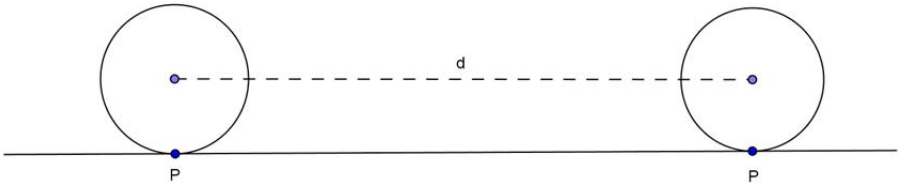
38) A pick-up truck is fitted with new tires which have a diameter of 44 inches. How fast will the pick-up truck be moving when the wheels are rotating at 275 revolutions per minute? Express the answer in miles per hour rounded to the nearest whole number.

$\boxed{\text{A) 36 mph}}$ B) 31 mph C) 18 mph

D) 29 mph E) 24 mph

Solution: We need to determine the distance d the truck travels in 1 minute. In one revolution the center of the wheel will move a distance of πD where D is the diameter of the wheel (this is the distance the point of contact P will cover in one revolution, the length of the circumference of the wheel). Therefore, the total distance d traveled by the truck in 1 minute is given by

$d = 275 \times \pi D = 275 \times \pi \times 44 = 38013.27111$ inches



The velocity is then given by $v = \dfrac{38013.27111 \quad inches}{1 \quad \text{min}} = \dfrac{\dfrac{38013.27111}{12 \times 5280} \quad miles}{\dfrac{1}{60} \quad hour} = 35.99$ mph.

39) If $\sin\theta = a$, find the value of $\sin 3\theta$.

A) $3a - 4a^2$ B) $2a - 4a^3$ C) $3a + 4a^3$

D) $2a - 4a^2$ $\boxed{\text{E) } 3a - 4a^3}$

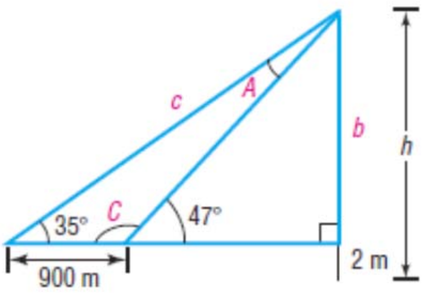
Solution: We have $\sin 3\theta = \sin(2\theta + \theta) = \sin 2\theta \cos\theta + \sin\theta \cos 2\theta = 2\sin\theta \cos^2\theta + \sin\theta(\cos^2\theta - \sin^2\theta)$, thus $\sin 3\theta = 2\sin\theta(1 - \sin^2\theta) + \sin\theta(1 - 2\sin^2\theta) = 3\sin\theta - 4\sin^3\theta$. Therefore, $\sin 3\theta = 3a - 4a^3$.

40) To measure the height of a mountain, a surveyor takes two sightings of the peak at a distance 900 meters apart on a direct line to the mountain. The first observation results in an angle of elevation of 47°, and the second results in an angle of elevation of 35°. If the transit is 2 meters high, what is the height *h* of the mountain?

[✓ A) $h = 1818$ meters]　　B) $h = 1817$ meters　　C) $h = 1816$ meters

D) $h = 1815$ meters　　E) Insufficient information to determine the height.

Solution: The situation is summarized in the figure below



Since C = 133° and A = 12°, using the law of sines we have that

$$c = \frac{a\sin C}{\sin A} = \frac{900\sin 133^\circ}{\sin 12^\circ} \approx 3165.86$$

Using the larger triangle we obtain that $\sin 35^\circ = \frac{b}{c}$, so

$b = 3165.86\sin 35^\circ \approx 1815.86$. Therefore, the height of the peak from ground level is approximately 1818 meters.

41) Let θ be an angle in the third quadrant such that $0 < \theta < 2\pi$. Find $\cos^{-1}(\cos\theta)$.

A) $\theta - \pi$　　[✓ B) $2\pi - \theta$]　　C) $\pi - \theta$　　D) θ　　E) $\theta - 2\pi$

Solution: The range of the function $\cos^{-1}$ is the interval $[0, \pi]$. The angle in this interval which has the same cos as the angle θ is $2\pi - \theta$.

42) The function $f(x) = \sin x$ is one to one when restricted to the interval $[\pi/2, 3\pi/2]$. Find its inverse on this interval.

A) $f^{-1}(x) = \sin^{-1}x - \pi$　　[✓ B) $f^{-1}(x) = \pi - \sin^{-1}x$]　　C) $f^{-1}(x) = \sin^{-1}x$

D) $f^{-1}(x) = \frac{1}{\sin x}$　　E) $f^{-1}(x) = \pi - \frac{1}{\sin x}$

Solution: It is easy to see from the graph of sine and arcsine that in order to get the inverse of $f(x)=\sin x$ on $[\pi/2, 3\pi/2]$, you need to reflect the graph arcsine about the x-axis and shift it up by π. Thus, we have $f^{-1}(x)=\pi-\sin^{-1}x$. Another solution is: let $F(x)=\sin x$ for $-\pi/2\le x\le \pi/2$ .Then, $y=f(x)=F(-x+\pi)$ on $\pi/2\le x\le 3\pi/2$. Therefore $F^{-1}(y)=-x+\pi$ which means that $x=\pi-F^{-1}(y)=\pi-\sin^{-1}y$. Thus, $f^{-1}(x)=\pi-\sin^{-1}x$.

43) Let d denote the length of a chord of a circle of radius R and let θ be the central angle form by the radii to the ends of the chord. Which of the following is the value of d in terms of R and θ?

A) $d=R\sqrt{2-\cos\theta}$

B) $d=R\sqrt{1-\cos\theta}$

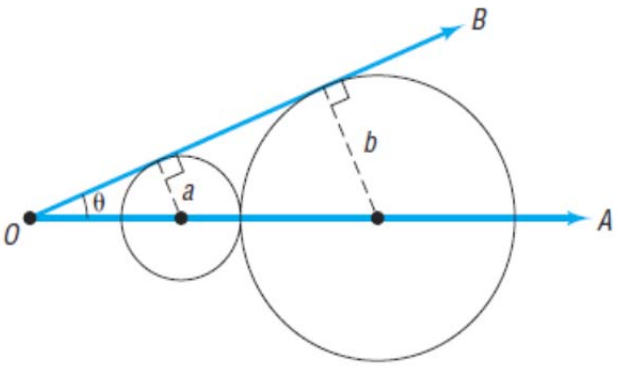
C) $d=R\sqrt{1+\cos\theta}$

$\boxed{\text{D) } d=2R\sin\left(\frac{\theta}{2}\right)}$

E) $d=4R\sin\left(\frac{\theta}{2}\right)$

Solution: The triangle defined by the center of the circle and the two ends of the chord has two sides of length R which form the angle θ and a third side, the chord, of length d. Applying the law of cosines to this triangle we get $d^2=2R^2-2R^2\cos\theta=4R^2\frac{(1-\cos\theta)}{2}=4R^2\sin^2\left(\frac{\theta}{2}\right)$. Hence, $d=2R\sin\left(\frac{\theta}{2}\right)$.

44) In the figure shown on the right, the smaller circle, whose radius is *a*, is tangent to the larger circle, whose radius is *b*. The ray $\overrightarrow{OA}$ contains a diameter of each circle, and the ray $\overrightarrow{OB}$ is tangent to each circle. If θ is the angle between the two rays, find $\cos\theta$.



A) $\cos\theta=\frac{\sqrt{ab}}{a+b}$

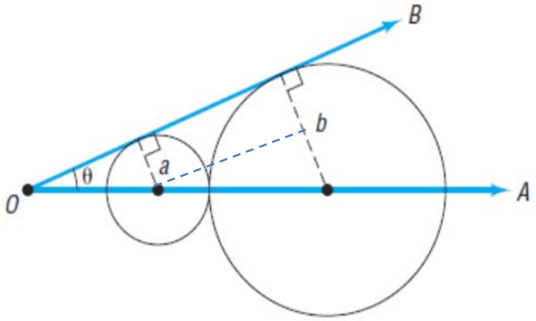
B) $\cos\theta=\frac{\sqrt{b-a}}{a+b}$

$\boxed{\text{C) } \cos\theta=\frac{2\sqrt{ab}}{a+b}}$

D) $\cos\theta=\frac{b-a}{a+b}$

E) $\cos\theta=\frac{ab}{a+b}$

Solution: Draw the right triangle as shown in the figure and notice that the hypotenuse is $a+b$, the short leg is $b-a$, and the other leg is $\sqrt{(a+b)^2-(b-a)^2}=2\sqrt{ab}$. Thus $\cos\theta=\dfrac{2\sqrt{ab}}{a+b}$



45) At a garden store, the weekly demand for Old English Roses is $q=420-12p$, where q represents the number of rose bushes that can be sold at the price p , in dollars, with $9.50\le p\le 20.00$. If rose bushes are currently selling for $15 each and they cost the store $2 each, what advice would you give the owner to maximize weekly profit?

A) Raise the price of each rose bush to $18.50.

B) Raise the price of each rose bush to $17.50.

C) Raise the price of each rose bush to $19.50.

D) Lower the price of each rose bush to $13.50.

E) Lower the price of each rose bush to $14.50.

Solution: The profit function, which is revenue minus cost, is given by the function $P(p)=q(p)\times p-2\times q(p)=(420-12p)(p-2)$. The maximum value of this function on the interval $9.50\le p\le 20.00$ is attained at $p=18.50$. Therefore the advice would be to raise the price of each rose bush to $18.50.

46) Let F_n denote the general term of the Fibonacci sequence defined recursively by $F_1=1$, $F_2=1$ and $F_n=F_{n-1}+F_{n-2}$ for all $n\in\mathbb{N}$. Find the determinant of the 2x2 matrix given by $\begin{bmatrix} F_{2016} & F_{2015} \\ F_{2015} & F_{2014} \end{bmatrix}$.

A) 1

B) -1

C) F_{2015}

D) $F^2{}_{2015}$

E) $F_{2015}F_{2016}$

Solution: Using the fact that $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{2015}=\begin{bmatrix} F_{2016} & F_{2015} \\ F_{2015} & F_{2014} \end{bmatrix}$ we get that the determinant is -1.

47) The figure below shows a circle of radius 2 centered at the origin on the *xy*-plane and two points $P(-\sqrt{3},-1)$ and $Q(\sqrt{3},-1)$ on the circle. Randomly select a third point R on the set that consists of the circle minus the points *P* and *Q*. Let α and β be the interior angles opposite to the vertex R of the triangle defined by the three points. Find the probability that $\alpha \geq \pi/4$ or $\beta \geq \pi/4$.

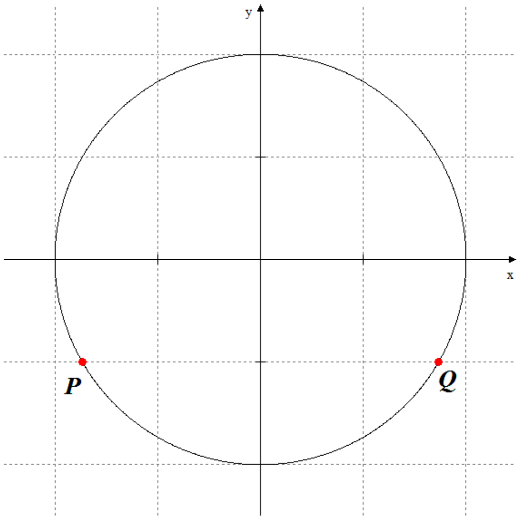
A) $\frac{1}{3}$

B) 1

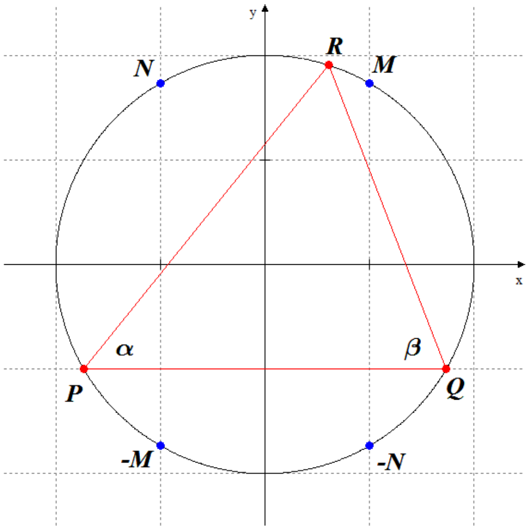
C) $\frac{3}{6}$

$\boxed{\text{D) } \frac{5}{6}}$

E) $\frac{1}{6}$



Solution: Let *A* be the event $\alpha \geq \pi/4$ and *B* the event $\beta \geq \pi/4$. We will use the addition rule $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ to compute the answer. Using the figure shown below, we can see that $\alpha \geq \pi/4$ when the third point *R* is on the arc from *M* to –*M* (traveling in the counterclockwise direction). This arc has length 2π. By symmetry, we can see that $\beta \geq \pi/4$ when the third point *R* is on the arc from *N* to –*N* (traveling in the clockwise direction). This arc also has length 2π. When *R* is a point on the arc from *M* to *N* (traveling in the counterclockwise direction), we have both, $\alpha \geq \pi/4$ and $\beta \geq \pi/4$. The length of this arc is $2\theta = 2\left(\frac{2\pi}{3} - \frac{\pi}{3}\right) = \frac{2\pi}{3}$, the central angle subtended by the arc times the radius of the circle. Since the total length of the circle is 4π we finally have



$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{2\pi}{4\pi} + \frac{2\pi}{4\pi} - \frac{2\pi/3}{4\pi} = \frac{5}{6}$$

Or, a shorter version of the solution is to consider the events –*A* and – *B*.

$$P(A \cup B) = 1 - P(-A \cap -B) = 1 - \frac{2\pi/3}{4\pi} = \frac{5}{6}$$

The event $-A \cap -B$ is given by the shortest arc from – *M* to –*N*.

48) Find, where defined, the derivative of $f(x) = \sin^{-1}(\sin x)$.

A) -1

B) 1

$\boxed{\text{C) } \dfrac{\cos x}{|\cos x|}}$

D) $\dfrac{\sin x}{|\cos x|}$

E) None of these

<u>Solution:</u> Applying the chain rule we get $f'(x) = \left(\sin^{-1}\right)'(\sin x) \times (\sin x)'$ which is

$$f'(x) = \frac{\cos x}{\sqrt{1-\sin^2 x}} = \frac{\cos x}{\sqrt{\cos^2 x}} = \frac{\cos x}{|\cos x|}$$

49) You need to buy some filing cabinets. You know that Cabinet X costs \$10 per unit, requires six square feet of floor space, and holds eight cubic feet of files. Cabinet Y costs \$20 per unit, requires eight square feet of floor space, and holds twelve cubic feet of files. You have been given \$140 for this purchase, though you don't have to spend that much. The office has room for no more than 72 square feet of cabinets. Let $x \geq 0$ be the number of model X cabinets purchased and let $y \geq 0$ be the number of model Y cabinets purchased. Which of the following mathematical models should be solved in order to answer the question: How many of each model should you buy, in order to maximize storage volume V?

A) $\begin{array}{l} \text{Maximize } V = 12x + 8y \\ \text{Subject to} \\ 10x + 20y \leq 140 \\ 6x + 8y \leq 72 \end{array}$

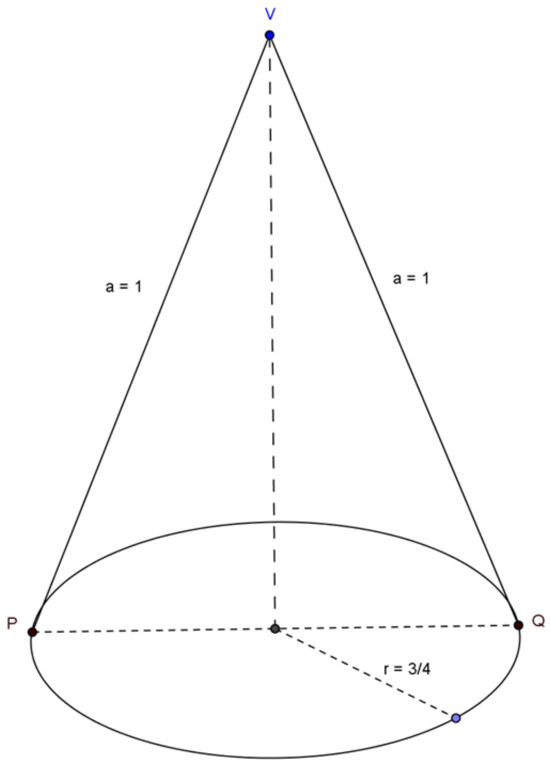
$\boxed{\text{B) } \begin{array}{l} \text{Maximize } V = 8x + 12y \\ \text{Subject to} \\ 10x + 20y \leq 140 \\ 6x + 8y \leq 72 \end{array}}$

C) $\begin{array}{l} \text{Maximize } V = 8x + 12y \\ \text{Subject to} \\ 10x + 20y < 140 \\ 6x + 8y < 72 \end{array}$

D) $\begin{array}{l} \text{Maximize } V = 8x + 12y \\ \text{Subject to} \\ 10x + 20y < 140 \\ 6x + 8y \leq 72 \end{array}$

E) $\begin{array}{l} \text{Maximize } V = 8x + 12y \\ \text{Subject to} \\ 20x + 10y \leq 140 \\ 8x + 6y \leq 72 \end{array}$

50) An infinite circular cone is cut by a plane perpendicular to its axis producing the circular cross section of radius $r = 3/4$ shown in the figure on the right. The points P and Q are diametrically opposed and at a distance $a = 1$ from the vertex V of the cone. Find the length of the shortest path on the surface of the cone from the point P to the point Q.



$\boxed{\text{A) } \sqrt{2+\sqrt{2}}}$

B) $\dfrac{3\pi}{4}$

C) $3/2$

D) $\sqrt{2-\sqrt{2}}$

E) None of these

Solution: It is easy to see that if we cut the cone along the line joining V and Q, we can develop the cone into the circular sector of central angle $\dfrac{3\pi}{2}$ and radius 1 shown in the figure. The segments joining the center of the sector with the two copies of the point Q are identified, meaning that they are the same segment, joining V and Q, on the cone. Therefore, by symmetry, the point P is given by $P\left(-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$. The distance between P and Q on the cone is the same as the distance between P and Q on the plane, namely, $\sqrt{2+\sqrt{2}}$. Therefore, the length of the shortest path on the cone from the point P to the point Q is $\sqrt{2+\sqrt{2}}$.