Forty First Annual Columbus State University
Invitational Mathematics Tournament

Sponsored by
The Columbus State University
Department of Mathematics
March 7, 2015
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The Columbus State University Mathematics faculty welcomes you to this year's tournament and to our campus. We wish you success on this test and in your future studies.

Instructions

This is a 90-minute, 50-problem, multiple-choice exam. There are five possible responses to each question. You should select the one "best" answer for each problem. In some instances this may be the closest approximation rather than an exact answer. You may mark on the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer circle the choice you have made on the test booklet. Carefully transfer your answers to the score sheet. Completely darken the blank corresponding to the letter of your response to each question. Mark your answer boldly with a No. 2 pencil. If you must change an answer, completely erase the previous choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response. The examination will be scored on the basis of +12 for each correct answer, −3 for each incorrect selection, and 0 for each omitted item. Each student will be given an initial score of +200.

Pre-selected problems will be used as tie-breakers for individual awards. These problems, designated with an asterisk (*), in order of consideration are: 5, 10, 11, 12, 14, 17, 20, 21, 23, 26, 27, 28, 30, 31, 32, 35, 37, 41, 47, and 48.

Throughout the exam, $\overline{AB}$ will denote the line segment from point A to point B and AB will denote the length of $\overline{AB}$. Pre-drawn geometric figures are not necessarily drawn to scale. The measure of the angle $\angle ABC$ is denoted by $m\angle ABC$.

Review and check your score sheet carefully. Your student identification number and your school number must be encoded correctly on your score sheet.

When you complete your test, bring your pencil, scratch paper and answer sheet to the test monitor. Leave the room after you have handed in your answer sheet. Please leave quietly so as not to disturb the other contestants. Do not congregate outside the doors by the testing area. You may keep your copy of the test. Your sponsor will have a copy of solutions to the test problems.

Do not open your test until instructed to do so!

1) What is the 2015th digit (to the right of decimal point) in the decimal expansion of $\dfrac{224}{1111}$?

A) 0 B) 1 C) 2 D) 5 E) 6

2) If the three zeros of the polynomial $p(x) = x^3 + bx^2 + 623x - 2015$ are positive integers, what is the value of b?

A) 33 B) -33 C) -49 D) 49 E) 142

3) What is the largest number of points in which the graphs of a third degree polynomial and a fifth degree polynomial can meet?

A) 3 B) 5 C) 8 D) 2 E) 15

4) Which of the following numbers is not a divisor of 2015^5-2015?

A) 16 B) 32 C) 256 D) 512 E) Both, C) and D)

5) What is the number of ordered pairs (x, y) of positive integers that satisfy the equation $31x + 13y = 2015$?

A) 0 B) 4 C) 5 D) 6 E) Infinitely many pairs.

6) Find the sum of the solutions of the equation $\sqrt{2x+7} + 4 = x$.

A) 9 B) -9 C) 10 D) -10 E) None of these

7) At the end of the 1998 season, the National Football League's all-time leading passer during regular season play was Dan Marino with 4763 completed passes out of 7989 attempts. In his debut 1998 season, Peyton Manning made 326 completed passes out of 575 attempts. What is the smallest number of consecutive completed passes that Peyton Manning would have to make to exceed Dan Marino's pass completion percentage?

A) 41.7 B) 42 C) 42.5 D) 43 E) None of these

8) On Tuesday the price of gas was 5% more expensive that on Monday. On Wednesday the price went down by 2%. If a gallon of gas cost $ 3.59 on Wednesday, how much did it cost on Monday? Please, round your answer to two decimal places.

A) $3.35 B) $2.44 C) $3.40 D) $3.48 E) $ 3.49

9) Let $\Re = \{x \in \mathbb{R} : x \neq -1\}$ and for all $x, y \in \Re$ define $x \square y = x + y + xy$. Find all solutions in $\Re$ of the equation $x \square x = 2 \square x$.

A) $x = 0$ B) $x = 0$ and $x = 2$ C) $x = -1$ and $x = 2$

D) $x = 2$ E) $x = 0$, $x = -1$ and $x = 2$

10) How many integers between 1 and 2015 are multiples of 3 or 4?

A) 1008 B) 1174 C) 1006 D) 1007 E) 1176

11) An island has three kinds of inhabitants, knights, who always tell the truth, knaves, who always lie, and spies who can either lie or tell the truth. You encounter three people A, B, and C. You know one of these people is a knight, one is a knave, and one is a spy. A says "C is the knave", B says "A is the knight", and C says "I am the spy". Which of the following is true?

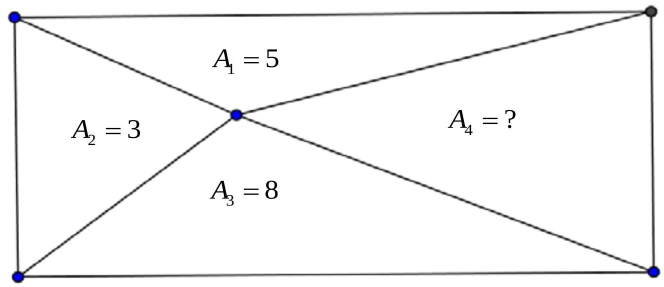
A) A is the knight B) B is the knight C) C is the knight

D) There is more than one solution E) There is no solution

12) In the rectangle shown in the figure, the area of the three the triangles are $A_1 = 5$, $A_2 = 3$, and $A_3 = 8$. Find the area A_4 of the remaining triangle.

A) 7 B) 8 C) 9

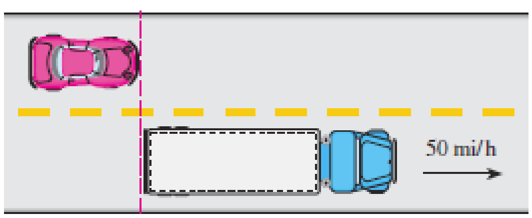
D) 10 D) 11



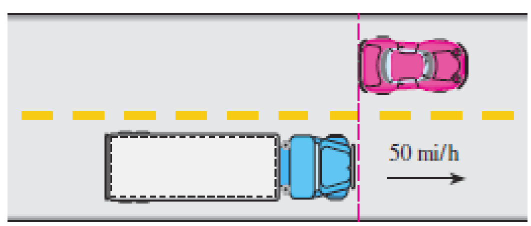
13) Find the remainder of $1006^{2015} + 1007^{2015} + 2016$ when divided by 2013.

A) 2014 B) 2015 C) 2 D) 3 E) 4

14) A woman driving a car 14 feet long is passing a truck 30 feet long. The truck is traveling at 50 mi/h. How fast must the woman drive her car so that she can pass the truck completely in 6 seconds, from the position shown in figure (a) to the position shown in figure (b)? Please, round your answer to two decimal places.



(a)



(b)

A) 69.1 mi/h B) 50.68 mi/h C) 51 mi/h D) 59.55 mi/h E) 51.59 mi/h

15) Find the center (h,k) of the circle on the *xy*-plane, passing through the points $(0,b)$, $(-a,0)$ and $(a,0)$.

A) $(h,k)=\left(0,\dfrac{a^2-b^2}{2b}\right)$ B) $(h,k)=\left(0,\dfrac{b^2+a^2}{2b}\right)$ C) $(h,k)=\left(0,\dfrac{b^2-a^2}{b}\right)$

D) $(h,k)=\left(\dfrac{a^2-b^2}{2b},0\right)$ E) $(h,k)=\left(0,\dfrac{b^2-a^2}{2b}\right)$

16) A driver sets out on a journey. For the first half of the distance she drives at the leisurely pace of 30 mi/h; during the second half she drives 60 mi/h. What is her average speed on this trip?

A) 35 mi/h B) 40 mi/h C) 45 mi/h

D) 50 mi/h E) 55 mi/h

17) It takes a paddle boat 50 minutes to travel 4 miles up a river and 4 miles back, going at a steady speed of 10 miles per hour (with respect to the water). Find the speed of the current in miles per hour.

A) 2 mph B) 9.92 mph C) 2.90 mph

D) 3 mph E) None of these

18) Let $a > b > 0$, $a^2 + b^2 = 3ab$. Find the value of $\dfrac{a+b}{a-b}$.

A) $\sqrt{2}$ B) $\sqrt{3}$ C) $\sqrt{5}$ D) 2 E) 3

19) If $\sqrt{17-12\sqrt{2}} = a + b\sqrt{2}$, which of the following is $\left(a+b\sqrt{2}\right)^{-1}$?

A) $17+2\sqrt{2}$ B) $\dfrac{1}{17-2\sqrt{2}}$ C) $3-2\sqrt{2}$

D) $3+2\sqrt{2}$ E) $2-3\sqrt{2}$

20) Find all real solutions of the inequality $x^2 - 3 \le \left|3 - x^2\right|$

A) $-\sqrt{3} < x < \sqrt{3}$ B) $-\sqrt{3} \le x \le \sqrt{3}$ C) $-\infty < x < \infty$

D) $-3 < x < 3$ E) $-3 \le x \le 3$

21) If the function $f(x) = \dfrac{1}{x^2 + 2x + c}$ is defined for all real numbers, then find all possible values of c.

A) $c > 1$ B) $c = 1$ C) $c < 1$ D) $c \le 1$ E) $c > 2$

22) Which of the following is the equation of the circle that goes through the origin and is tangent to the line $-x+y=8$ at the point $(0,8)$?

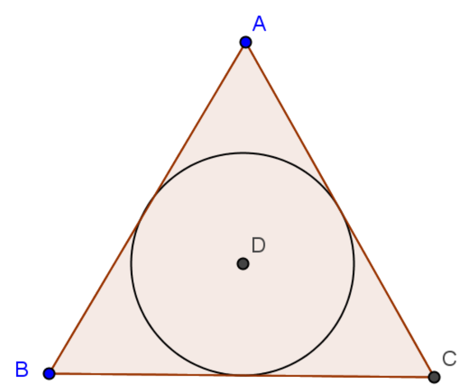
A) $(x+4)^2+(y+4)^2=\sqrt{32}$ B) $(x+4)^2+(y+4)^2=32$ C) $(x-4)^2+(y-4)^2=16$

D) $(x-4)^2+(y-4)^2=\sqrt{32}$ E) $(x-4)^2+(y-4)^2=32$

23) Suppose that $r_1, r_2, \ldots, r_{2015}$ are the 2015 roots of the polynomial $p(x)=x^{2015}+2014x+2016$. Find the average of $r_1^{2015}, r_2^{2015}, \ldots, r_{2015}^{2015}$.

A) -2014   B) 2014   C) 2016   D) -2016   E) 2015

24) The area of the equilateral triangle shown in the figure on the right is 6. Find the area of the inscribed circle.



A) $2\pi/\sqrt{3}$ B) 2π C) $\sqrt{3}\pi$

D) $\pi/\sqrt{3}$ E) $\sqrt{3}\pi/2$

25) What is the smallest value of $x+y$ if x and y are positive integers such that

$$\frac{1}{x}+\frac{1}{2x}+\frac{1}{3x}=\frac{1}{y^2-2y}?$$

A) 6   B) 44   C) 50   D) 56   E) 82

26) The straight line $\frac{x}{4}+\frac{y}{3}=1$ and the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ intersect at two points A and B. A third point P is selected on the ellipse in such way that ΔPAB has area 4. How many points P on the ellipse can be selected in the same way?

A) 1   B) 2   C) 3   D) 4   E) Infinitely many points

27) There are five different products with the same price. Beginning today, every day each product will get either a 10% or 20% discount. Let r be the ratio of the highest price to the lowest price. On the days in which the five prices are all different, what is the minimum value of r ?

A) $\left(\frac{9}{8}\right)^3$ B) $\left(\frac{9}{8}\right)^4$ C) $\left(\frac{9}{8}\right)^5$ D) $\frac{9}{8}$ E) $\left(\frac{9}{8}\right)^2$

28) Let x, y be positive integers so that $\sqrt{x-7}+\sqrt{x+2}=y$. What is the value of y ?

A) 9 B) 3 C) 1 D) 5 E) 6

29) There are two positive solutions to the equation $\log_{2x} 2+\log_4 2x=-\frac{3}{2}$. What is the product of the two solutions?

A) $1/21$ B) $3/21$ C) $1/32$ D) $1/8$ E) 2

30) Let $a>0$, $a\neq 1$, and $\frac{p}{q}\neq 0$ a rational number. Which of the following functions is equal to the function $f(x)=\log_{a^{p/q}} x$, $x>0$?

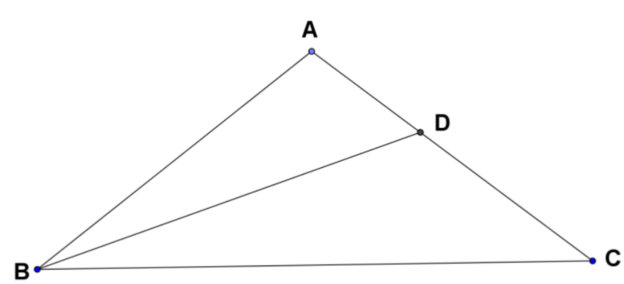
A) $\log_a x^{p/q}$ B) $-\frac{p}{q}\log_a x$ C) $\log_a x^{-q/p}$

D) $\log_a x^{q/p}$ E) $\frac{p}{q}\log_a x$

31) The isosceles triangle ΔABC shown in the figure has the property that $\angle BAC=100^{\circ}$, $AB=AC$ and BD is the bisector of the angle $\angle ABC$. If $BD=7.5$ and $AD=2.5$, find BC.

A) 10 B) 9.5 C) 18.75

D) 11.5 E) None of these



32) It is claimed that 40% of Americans do not have any health insurance. In a randomly selected group of three people, what is the probability that only one of them has health insurance?

A) 0.096 B) 0.288 C) 0.712

D) 0.144 E) 0.432

33) Sue owns 5 different skirts, 7 different blouses, and 6 different pairs of slacks. The only combination she will not wear is her orange blouse with her pink skirt or her blue slacks. Beginning on the first day of school, she decides to wear a different outfit on each school day as long as possible. On what day must she finally wear an outfit that she has already worn?

A) 77th B) 76th C) 74th D) 75th E) None of these

34) 12 points lie in a plane in such a way that exactly 5 of the points are on one straight line and apart from these 5 points, no three points lie on a straight line. Find the total number of distinct triangles that can be drawn with vertices on the 12 points.

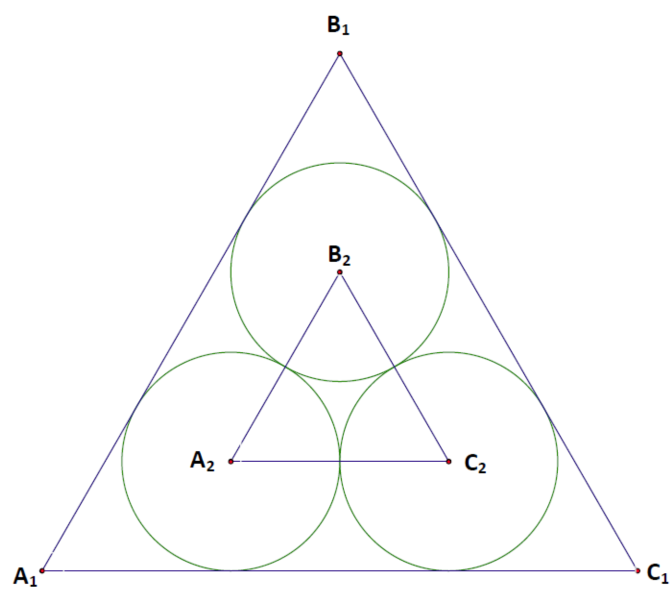
A) 105 B) 175 C) 210 D) 455 E) None of these

35) Let S_1 denote the area of the equilateral triangle $\Delta A_1B_1C_1$ and S_2 the area of the equilateral triangle $\Delta A_2B_2C_2$ whose vertices are the centers of the three largest circles of the same radius tangent to each other and the sides of $\Delta A_1B_1C_1$ as shown in the figure below. Find $\frac{S_2}{S_1}$.

A) $\frac{1}{1+\sqrt{3}}$ B) $\frac{1}{2\left(1+\sqrt{3}\right)}$

C) $\frac{1}{2\left(1+\sqrt{3}\right)^2}$ D) $\frac{1}{\left(1+\sqrt{3}\right)^2}$

E) $\frac{2}{\left(1+\sqrt{3}\right)^2}$



36) Consider the equilateral triangle $\Delta A_1B_1C_1$ shown on the picture below. Let A be the intersection of the angle bisectors. Construct the equilateral triangle $\Delta A_2B_2C_2$ whose vertices A_2, B_2, C_2 are the midpoints of the segments $\overline{AA_2}, \overline{AB_2}, \overline{AC_2}$, respectively. Construct now, inside the second triangle, a third triangle $\Delta A_3B_3C_3$ using the same method. Repeat this process 2015 times to get a sequence of triangles $\Delta A_iB_iC_i \quad i = 1, 2, 3, \ldots, 2015$. If the first triangle $\Delta A_1B_1C_1$ has perimeter 3, what is the sum of the perimeters of all the triangles in the sequence?

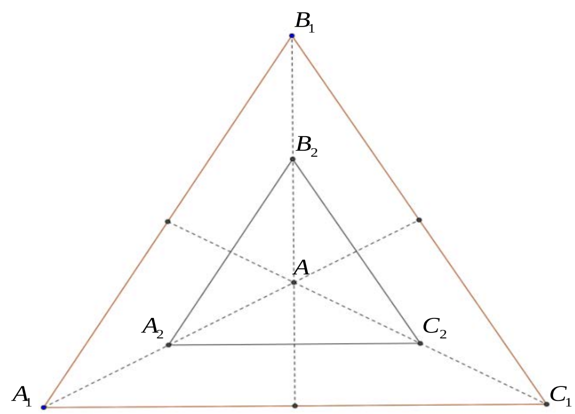
A) $\dfrac{6(2^{2016}-1)}{2^{2015}}$

B) $\dfrac{3(2^{2015}-1)}{2^{2014}}$

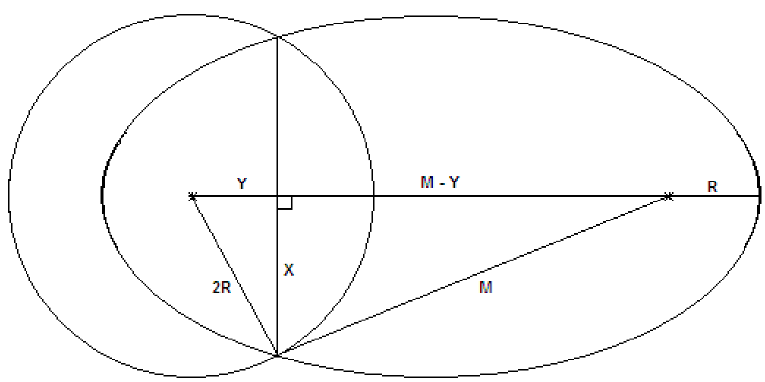
C) $\dfrac{3(2^{2016}-1)}{2^{2015}}$

D) $\dfrac{6(2^{2015}-1)}{2^{2014}}$

E) $\dfrac{3(2^{2015}-1)}{2^{2015}}$



37) Suppose that the distance between the foci determining a given ellipse is *M*, and the total distance from any point on the ellipse to the two foci is *M* + 2*R*. Suppose further that we have constructed, as shown in the figure below, a circle of radius 2*R* and having its center located at one of the two foci of the ellipse. Find the length *L* of the line segment that connects the two points of intersection of the ellipse and circle.



A) $L = \dfrac{4R\sqrt{M^2 - R^2}}{M}$

B) $L = \dfrac{R\sqrt{M^2 - R^2}}{M}$

C) $L = \dfrac{2R\sqrt{M^2 - R^2}}{M}$

D) $L = \dfrac{4R\sqrt{M^2 + R^2}}{M}$

E) $L = \dfrac{2\sqrt{M^2 + R^2}}{M}$

38) A pick-up truck is fitted with new tires which have a diameter of 44 inches. How fast will the pick-up truck be moving when the wheels are rotating at 275 revolutions per minute? Express the answer in miles per hour rounded to the nearest whole number.

A) 36 mph B) 31 mph C) 18 mph

D) 29 mph E) 24 mph

39) If $\sin\theta = a$, find the value of $\sin 3\theta$.

A) $3a - 4a^2$ B) $2a - 4a^3$ C) $3a + 4a^3$

D) $2a - 4a^2$ E) $3a - 4a^3$

40) To measure the height of a mountain, a surveyor takes two sightings of the peak at a distance 900 meters apart on a direct line to the mountain. The first observation results in an angle of elevation of 47°, and the second results in an angle of elevation of 35°. If the transit is 2 meters high, what is the height *h* of the mountain?

A) $h = 1818$ meters B) $h = 1817$ meters C) $h = 1816$ meters

D) $h = 1815$ meters E) Insufficient information to determine the height.

41) Let θ be an angle in the third quadrant such that $0 < \theta < 2\pi$. Find $\cos^{-1}(\cos\theta)$.

A) $\theta - \pi$ B) $2\pi - \theta$ C) $\pi - \theta$ D) θ E) $\theta - 2\pi$

42) The function $f(x) = \sin x$ is one to one when restricted to the interval $[\pi/2, 3\pi/2]$. Find its inverse on this interval.

A) $f^{-1}(x) = \sin^{-1} x - \pi$ B) $f^{-1}(x) = \pi - \sin^{-1} x$ C) $f^{-1}(x) = \sin^{-1} x$

D) $f^{-1}(x) = \dfrac{1}{\sin x}$ E) $f^{-1}(x) = \pi - \dfrac{1}{\sin x}$

43) Let d denote the length of a chord of a circle of radius R and let θ be the central angle form by the radii to the ends of the chord. Which of the following is the value of d in terms of R and θ?

A) $d = R\sqrt{2-\cos\theta}$

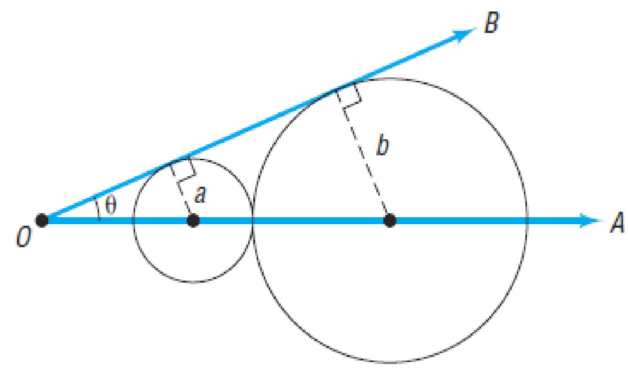
B) $d = R\sqrt{1-\cos\theta}$

C) $d = R\sqrt{1+\cos\theta}$

D) $d = 2R\sin\left(\frac{\theta}{2}\right)$

E) $d = 4R\sin\left(\frac{\theta}{2}\right)$

44) In the figure shown on the right, the smaller circle, whose radius is a, is tangent to the larger circle, whose radius is b. The ray $\overrightarrow{OA}$ contains a diameter of each circle, and the ray $\overrightarrow{OB}$ is tangent to each circle. If θ is the angle between the two rays, find $\cos\theta$.



A) $\cos\theta = \frac{\sqrt{ab}}{a+b}$

B) $\cos\theta = \frac{\sqrt{b-a}}{a+b}$

C) $\cos\theta = \frac{2\sqrt{ab}}{a+b}$

D) $\cos\theta = \frac{b-a}{a+b}$

E) $\cos\theta = \frac{ab}{a+b}$

45) At a garden store, the weekly demand for Old English Roses is $q = 420 - 12p$, where q represents the number of rose bushes that can be sold at the price p, in dollars, with $9.50 \le p \le 20.00$. If rose bushes are currently selling for \$15 each and they cost the store \$2 each, what advice would you give the owner to maximize weekly profit?

A) Raise the price of each rose bush to \$18.50.

B) Raise the price of each rose bush to \$17.50.

C) Raise the price of each rose bush to \$19.50.

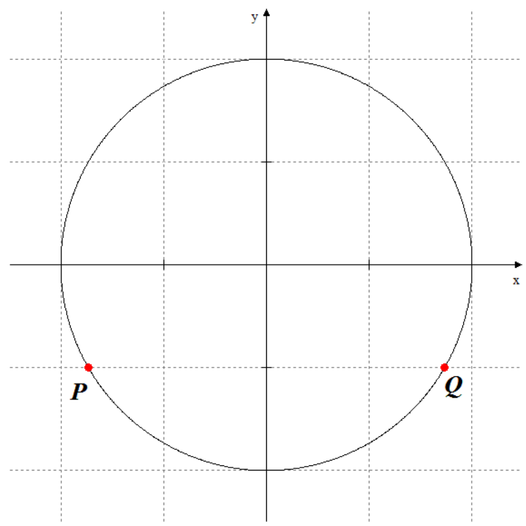
D) Lower the price of each rose bush to \$13.50.

E) Lower the price of each rose bush to \$14.50.

46) Let F_n denote the general term of the Fibonacci sequence defined recursively by $F_1 = 1$, $F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \in \mathbb{N}$. Find the determinant of the 2x2 matrix given by $\begin{bmatrix} F_{2016} & F_{2015} \\ F_{2015} & F_{2014} \end{bmatrix}$.

A) 1 B) -1 C) F_{2015} D) $F^2{}_{2015}$ E) $F_{2015}\,F_{2016}$

47) The figure below shows a circle of radius 2 centered at the origin on the *xy*-plane and two points $P(-\sqrt{3},-1)$ and $Q(\sqrt{3},-1)$ on the circle. Randomly select a third point R on the set that consists of the circle minus the points *P* and *Q*. Let α and β be the interior angles opposite to the vertex R of the triangle defined by the three points. Find the probability that $\alpha \geq \pi/4$ or $\beta \geq \pi/4$.



A) $\frac{1}{3}$ B) 1 C) $\frac{3}{6}$

D) $\frac{5}{6}$ E) $\frac{1}{6}$

48) Find, where defined, the derivative of $f(x) = \sin^{-1}(\sin x)$.

A) -1 B) 1 C) $\dfrac{\cos x}{|\cos x|}$ D) $\dfrac{\sin x}{|\cos x|}$ E) None of these

49) You need to buy some filing cabinets. You know that Cabinet X costs $10 per unit, requires six square feet of floor space, and holds eight cubic feet of files. Cabinet Y costs $20 per unit, requires eight square feet of floor space, and holds twelve cubic feet of files. You have been given $140 for this purchase, though you don't have to spend that much. The office has room for no more than 72 square feet of cabinets. Let $x \geq 0$ be the number of model X cabinets purchased and let $y \geq 0$ be the number of model Y cabinets purchased. Which of the following mathematical models should be solved in order to answer the question: How many of each model should you buy, in order to maximize storage volume *V*?

A) Maximize $V = 12x + 8y$
Subject to
$10x + 20y \le 140$
$6x + 8y \le 72$

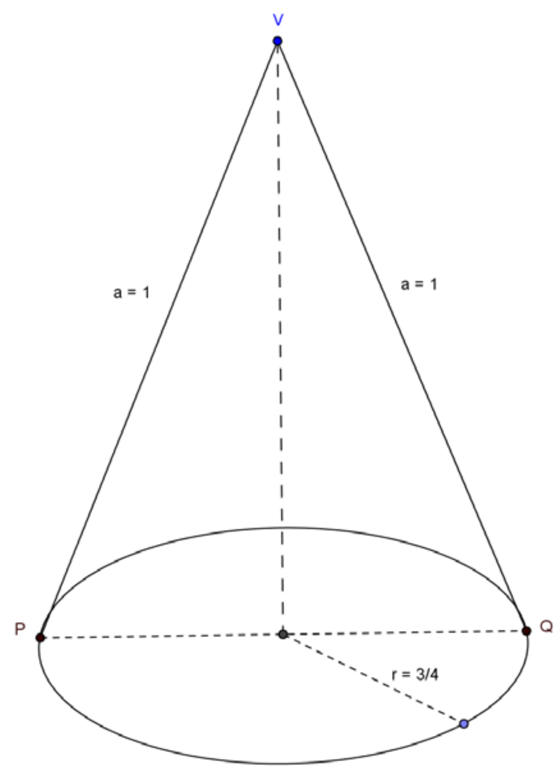
B) Maximize $V = 8x + 12y$
Subject to
$10x + 20y \le 140$
$6x + 8y \le 72$

C) Maximize $V = 8x + 12y$
Subject to
$10x + 20y < 140$
$6x + 8y < 72$

D) Maximize $V = 8x + 12y$
Subject to
$10x + 20y < 140$
$6x + 8y \le 72$

E) Maximize $V = 8x + 12y$
Subject to
$20x + 10y \le 140$
$8x + 6y \le 72$

50) An infinite circular cone is cut by a plane perpendicular to its axis producing the circular cross section of radius $r = 3/4$ shown in the figure on the right. The points *P* and *Q* are diametrically opposed and at a distance $a = 1$ from the vertex *V* of the cone. Find the length of the shortest path on the surface of the cone from the point *P* to the point *Q*.



A) $\sqrt{2+\sqrt{2}}$ B) $\frac{3\pi}{4}$ C) $3/2$

D) $\sqrt{2-\sqrt{2}}$ E) None of these