SOME QUESTIONS ABOUT EQUILATERAL TRIANGLES IN $\mathbb{Z}^4$

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Abstract. We are aiming for a characterization of all three points in $\mathbb{R}^4$ with integer coordinates which are at the same Euclidean distance apart. Some questions are proposed.

1. Description

Question 1: If we start with an arbitrary point A (different of the origin) in $\mathbb{Z}^4$ can we find a point B such that the triangle OAB is an irreducible equilateral triangle ?

Question 2: Are there finitely many families covering all possible equilateral triangles in $\mathbb{Z}^4$?

Question 3: Does every equilateral triangle in $\mathbb{Z}^4$ extend to a regular tetrahedron ?

Question 4: *Given a natural odd number k and six integer values $\Delta_{12} \geq 0$, Δ_{34}, $\Delta_{13} \geq 0$, Δ_{24}, Δ_{14} and $\Delta_{23} > 0$ such that,*

$$(1) \qquad \begin{cases} 3k^2 = (\Delta_{12} - \Delta_{34})^2 + (\Delta_{13} + \Delta_{24})^2 + (\Delta_{23} - \Delta_{14})^2 \\ \Delta_{12}\Delta_{34} - \Delta_{13}\Delta_{24} + \Delta_{14}\Delta_{23} = 0, \end{cases}$$

is there a family of equilateral triangles OPQ with P and Q living in the "two" dimensional space $\mathcal{S}$ of all vectors in $(u, v, w, t) \in \mathbb{Z}^4$, such that

$$(2) \qquad \begin{cases} (0)u + \Delta_{34}v + \Delta_{24}w + \Delta_{23}t = 0 \\ \Delta_{23}u + \Delta_{13}v + \Delta_{12}w + (0)t = 0? \end{cases}$$

If the answer to this question is "Yes", how can one find a minimal triangle which generates it? What are its side-lengths ?

References

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